Towards the Quantum Electrodynamics on the Poincaré Group

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Abstract

A general scheme of construction and analysis of physical fields on the various homogeneous spaces of the Poincaré group is presented. Different parametrizations of the field functions and harmonic analysis on the homogeneous spaces are studied. It is shown that a direct product of Minkowski spacetime and two-dimensional complex sphere is the most suitable homogeneous space for the subsequent physical applications. The Lagrangian formalism and field equations on the Poincaré group are considered. A boundary value problem for the relativistically invariant system is defined. General solutions of this problem are expressed via an expansion in hyperspherical harmonics on the complex two-sphere. A physical sense of the boundary conditions is discussed. The boundary value problems of the same type are studied for the Dirac and Maxwell fields. In turn, general solutions of these problems are expressed via convergent Fourier type series. Field operators, quantizations, causal commutators and vacuum expectation values of time ordered products of the field operators are defined for the Dirac and Maxwell fields, respectively. Interacting fields and inclusion of discrete symmetries into the framework of quantum electrodynamics on the Poincaré group are discussed.

Keywords: fields on the Poincaré group, harmonic analysis, boundary value problem, relativistic wave equations, quantization

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1 Introduction

An unification of spacetime and internal symmetries of elementary particles is a long standing problem in physics. It is well known that the standard approach in this area is a search of some unification group which includes the Poincaré group $\mathcal{P}$ and the group (or, groups) of internal symmetries ($SU(2)$, $SU(3)$ and so on) as a subgroup [8]. However, in this approach a physical sense of the unification group is unclear. The most natural way for solving this problem was proposed by Finkelstein [32], he showed that elementary particles models with internal degrees of freedom can be described on manifolds larger then Minkowski spacetime (homogeneous spaces of the Poincaré group). The quantum field theories on the Poincaré group were discussed in papers [69, 58, 15, 4, 63, 111, 70, 28, 42, 46].

A consideration of the field models on the homogeneous spaces leads naturally to a generalization of the concept of wave function (fields on the Poincaré group). The general form

of these fields closely relates with the structure of the Lorentz and Poincaré group representations [36, 78, 13, 42] and admits the following factorization $f(x,\mathbf{z}) = \phi^n(\mathbf{z})\psi_n(x)$, where $x \in T_4$ and $\phi^n(\mathbf{z})$ form a basis in the representation space of the Lorentz group. Since the functions $f(x,\mathbf{z})$ are considered as a starting point of the present research, then it is very important to find a correct parametrization for these functions. The parametrization of the field functions is important also for the subsequent tasks of harmonic analysis and solutions of relativistic wave equations. The most natural way to do it is to express the functions $f(x,\mathbf{z})$ via the matrix elements of the Poincaré group. The functions $\psi_n(x)$ can be expressed via the exponentials. As is known, exponentials define unitary representations of the translation subgroup T_4. In turn, the functions $\phi^n(\mathbf{z})$ can be parametrized in the form of matrix elements of the Lorentz group. Matrix elements of irreducible representations of the Lorentz group were studied in the works [25, 26, 27, 29, 30, 44, 108, 109, 96, 118, 60, 50, 106, 51], where various expressions for these elements are found, however, in rather complicate and cumbersome form. The most simple form of the matrix elements for spinor representations of the Lorentz group has been found in the works [114, 117] in the basis of complex angular momentum, that corresponds to a local isomorphism $SL(2,C) \sim SU(2)\otimes SU(2)$. In essence, this form of the matrix elements presents itself a four-dimensional analog of Legendre spherical functions (hyperspherical functions). Due to this form a relationship between matrix elements of the Lorentz group and special functions becomes more clear. Moreover, it allows us to investigate the functions $f(x,\mathbf{z})$ within the framework of the powerful mathematical theory presented in the works of Vilenkin, Klimyk, Miller and Talman [120, 76, 110, 59, 121]. In the sections 2–4 of the present work we study various forms of matrix elements of the groups $SU(2)$, $SU(1,1)$ and $SL(2,\mathbb{C})$ both for finite- and infinite-dimensional representations. In parallel, we consider basic facts concerning harmonic analysis on these groups. As a rule, harmonic analysis on the noncompact groups is a very complicate mathematical theory and by this reason physicists hardly used this mathematics (obviously, it is one of the main obstacles that decelerate development of the quantum field theory on the Poincaré group). Therefore, it is very important to accommodate the abstract harmonic analysis to physics, that is, to construct a physically meaningful theory. With this end in view we restrict ourselves mainly by a consideration of finite dimensional representations, since they lead to local physical fields on the homogeneous spaces. Due to this fact, all the basic physical quantities, such as solutions of relativistic wave equations, field operators, causal commutators and so on, can be expressed via Fourier-type series on the homogeneous spaces.

The following logical step consists in definition of the Lagrangian formalism and field equations on the various homogeneous spaces of $\mathcal{P}$. The field equations for arbitrary spin are derived by the standard variation procedure from a selected Lagrangian. The use of harmonic analysis on the homogeneous spaces allows us to set up a boundary value problem for relativistically invariant system. It is shown that solutions of this problem are expressed via Fourier series on the two-dimensional complex sphere. In the sections 8 and 9 we study boundary value problems of the same type for the Dirac and Maxwell fields, respectively. The representation of the Dirac and Maxwell fields as fields on the Poincaré group allows us to consider these fields on an equal footing, from the one group theoretical viewpoint. Moreover, a definition of the Maxwell field on the complex two-sphere leads naturally to a Riemann-Silberstein representation for the electromagnetic field [124, 104, 12] (or Majorana-Oppenheimer formulation of quantum electrodynamics). Such basic notions of quantum field theory as field operators, quantization, causal commutators, vacuum expectation values of time ordered products have been defined for the Dirac and Maxwell fields in the sections 8

and 9. Interacting fields are discussed in the section 10.

2 The group $SU(2)$

In this section we briefly consider some basic facts concerning the group $SU(2)$ and its representations. The group $SU(2)$ is an universal covering of the three-dimensional rotation group $SO(3)$. Any matrix u from $SU(2)$ has the form

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \tag{1}$$

where $\det u = 1$ and, therefore, $|\alpha|^2 + |\beta|^2 = 1$. An Euler parameterization of (1) has the following form:

$$u = \begin{pmatrix} \cos\frac{\theta}{2} e^{\frac{i(\varphi+\psi)}{2}} & i\sin\frac{\theta}{2} e^{\frac{i(\varphi-\psi)}{2}} \\ i\sin\frac{\theta}{2} e^{\frac{i(\psi-\varphi)}{2}} & \cos\frac{\theta}{2} e^{-\frac{i(\varphi+\psi)}{2}} \end{pmatrix}, \tag{2}$$

where $0 \le \varphi \le 2\pi$, $0 < \theta < \pi$, $-2\pi \le \varphi < 2\pi$. It is easy to verify that from (2) it follows directly a decomposition

$$u(\varphi, \theta, \psi) = \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{\frac{i\psi}{2}} & 0 \\ 0 & e^{-\frac{i\psi}{2}} \end{pmatrix}. \tag{3}$$

The formula (3) defines *a Cartan decomposition* $G = KAK$ for the group $G = SU(2)$.

The group $SU(2)$ has three one-parameter subgroups

$$\omega_1(t) = \begin{pmatrix} \cos\frac{t}{2} & i\sin\frac{t}{2} \\ i\sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}, \quad \omega_2(t) = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}, \quad \omega_3(t) = \begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix}.$$

The tangent matrices A_i of these subgroups have the form

$$\begin{aligned} A_1 &= \left.\frac{d\omega_1(t)}{dt}\right|_{t=0} = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ A_2 &= \left.\frac{d\omega_2(t)}{dt}\right|_{t=0} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ A_3 &= \left.\frac{d\omega_3(t)}{dt}\right|_{t=0} = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

The elements A_i form a basis of Lie algebra $\mathfrak{su}(2)$ and satisfy the relations

$$[A_1, A_2] = A_3, \quad [A_2, A_3] = A_1, \quad [A_3, A_1] = A_2.$$

An Euler parameterization of infinitesimal operators A_i has the form:

$$\mathsf{A}_1 = \cos\psi\frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\varphi} - \cot\theta\sin\psi\frac{\partial}{\partial\psi}, \tag{4}$$

$$\mathsf{A}_2 = -\sin\psi\frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\varphi} - \cot\theta\cos\psi\frac{\partial}{\partial\psi}, \tag{5}$$

$$\mathsf{A}_3 = \frac{\partial}{\partial\psi}. \tag{6}$$

Since $SU(2)$ is a compact group, then all its representations are finite-dimensional. They are realized in symmetric representation spaces Sym_k ($k=2l$) of polynomials

$$p(z_0, z_1) = \sum_{(\alpha_1,\ldots,\alpha_k)} \frac{1}{k!} a^{\alpha_1\cdots\alpha_k} z_{\alpha_1}\cdots z_{\alpha_k},$$

The dimension of Sym_k is equal to $2l+1$. Operators $T_l(u)$, $u\in SU(2)$, act in Sym_k via the formula

$$T_l(u)\varphi(\mathfrak{z}) = (\beta\mathfrak{z}+\bar{\alpha})^{2l}\varphi\left(\frac{\alpha\mathfrak{z}+\bar{\beta}}{\beta\mathfrak{z}+\bar{\alpha}}\right),$$

where $\mathfrak{z}=z_0/z_1$. The matrix element $t^l_{mn} = e^{-i(m\varphi+n\psi)}\langle T_l(\theta)\psi_n, \psi_m\rangle$ of the group $SU(2)$ in the polynomial basis

$$\psi_n(\mathfrak{z}) = \frac{\mathfrak{z}^{l-n}}{\sqrt{\Gamma(l-n+1)\Gamma(l+n+1)}}, \quad -l\le n\le l, \tag{7}$$

where

$$T_l(\theta)\psi(\mathfrak{z}) = \left(i\sin\frac{\theta}{2}\mathfrak{z}+\cos\frac{\theta}{2}\right)^{2l}\psi\left(\frac{\cos\frac{\theta}{2}\mathfrak{z}+i\sin\frac{\theta}{2}}{i\sin\frac{\theta}{2}\mathfrak{z}+\cos\frac{\theta}{2}}\right),$$

has a form

$$\begin{aligned} t^l_{mn}(u) = e^{-i(m\varphi+n\psi)}\langle T_l(\theta)\psi_n,\psi_m\rangle = \\ \frac{e^{-i(m\varphi+n\psi)}\langle T_l(\theta)\mathfrak{z}^{l-n}\mathfrak{z}^{l-m}\rangle}{\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-n+1)\Gamma(l+n+1)}} = \\ e^{-i(m\varphi+n\psi)} i^{m-n}\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-n+1)\Gamma(l+n+1)}\times \\ \cos^{2l}\frac{\theta}{2}\tan^{m-n}\frac{\theta}{2}\times \\ \sum_{j=\max(0,n-m)}^{\min(l-n,l+n)} \frac{i^{2j}\tan^{2j}\dfrac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+n-j+1)\Gamma(m-n+j+1)}. \end{aligned} \tag{8}$$

Further, using the formula

$${}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\,\middle|\, z\right) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k\ge 0}\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)}\frac{z^k}{k!} \tag{9}$$

we can express the matrix element (8) via the hypergeometric function:

$$\begin{aligned} t^l_{mn}(u) = \frac{i^{m-n}e^{-i(m\varphi+n\psi)}}{\Gamma(m-n+1)}\sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}}\times \\ \cos^{2l}\frac{\theta}{2}\tan^{m-n}\frac{\theta}{2}\,{}_2F_1\left(\begin{matrix}m-l+1, 1-l-n\\ m-n+1\end{matrix}\,\middle|\, i^2\tan^2\frac{\theta}{2}\right), \end{aligned} \tag{10}$$

where $m \geq n$. At $m < n$ in the right part of (10) it needs to replace m and n by $-m$ and $-n$, respectively. Since l, m and n are finite numbers, then the hypergeometric series is interrupted.

The matrix elements (8) can be written in other form

$$t^l_{mn}(u) = i^{m-n} e^{-i(m\varphi+n\psi)} \sqrt{\frac{\Gamma(l-m+1)\Gamma(l-n+1)}{\Gamma(l+m+1)\Gamma(l+n+1)}} \times$$
$$\cot^{m+n}\frac{\theta}{2} \sum_{j=\max(m,n)}^{l} \frac{\Gamma(l+j+1)i^{2j}}{\Gamma(l-j+1)\Gamma(j-m+1)\Gamma(j-n+1)} \sin^{2j}\frac{\theta}{2}. \quad (11)$$

This form is derived from (8) by means of a factorization $u = kz$, where $k = \begin{pmatrix} \bar{\alpha}^{-1} & \beta \\ 0 & \bar{\alpha} \end{pmatrix}$ and $z = \begin{pmatrix} 1 & 0 \\ -\bar{\beta}/\bar{\alpha} & 1 \end{pmatrix}$. According to the Cartan decomposition (3), the matrix elements can be written as

$$\begin{aligned} t^l_{mn}(u) &= t^l_{mm}[u(\varphi,0,0)]t^l_{mn}(\theta)t^l_{nn}[u(0,0,\psi)] \\ &= e^{-i(m\varphi+n\psi)}t^l_{mn}(\theta), \end{aligned}$$

where

$$t^l_{mn}(\theta) = P^l_{mn}(\cos\theta),$$

here $P^l_{mn}(\cos\theta)$ is a generalized spherical function [36, 120]. Therefore, the formula (11) can be rewritten as follows

$$t^l_{mn}(u) = e^{-i(m\varphi+n\psi)}P^l_{mn}(\cos\theta). \quad (12)$$

The matrices $T_l(\theta)$ of irreducible representations of $SU(2)$ at $l = 0, \frac{1}{2}, 1$ have the following form:

$$T_0(\theta) = 1,$$

$$T_{\frac{1}{2}}(\theta) = \begin{pmatrix} P^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}} & P^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}} \\ P^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}} & P^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix},$$

$$T_1(\theta) = \begin{pmatrix} P^1_{-1-1} & P^1_{-10} & P^1_{-11} \\ P^1_{0-1} & P^1_{00} & P^1_{01} \\ P^1_{1-1} & P^1_{10} & P^1_{11} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{i\sin\theta}{\sqrt{2}} & -\sin^2\frac{\theta}{2} \\ \frac{i\sin\theta}{\sqrt{2}} & \cos\theta & \frac{i\sin\theta}{\sqrt{2}} \\ -\sin^2\frac{\theta}{2} & \frac{i\sin\theta}{\sqrt{2}} & \cos^2\frac{\theta}{2} \end{pmatrix}.$$

Using (4)–(6), it is easy to calculate the Laplace-Beltrami operator $\Delta = \mathsf{A}_1^2 + \mathsf{A}_2^2 + \mathsf{A}_3^2$ on the group $SU(2)$:

$$\Delta = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2} - 2\cos\theta\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right). \quad (13)$$

Matrix elements (12) of irreducible representations of the group $SU(2)$ are eigenfunctions of the operator (13):

$$\Delta t^l_{mn}(u) = -l(l+1)t^l_{mn}(u).$$

Substituting in this equality the expressions (13) and (12) instead Δ and $t^l_{mn}(u)$ and supposing $x=\cos\theta$, we come to the following differential equation

$$\left[(1-x^2)\frac{d^2}{dx^2}-2x\frac{d}{dx}-\frac{m^2+n^2-2mnx}{1-x^2}\right]P^l_{mn}(x)=-l(l+1)P^l_{mn}(x). \tag{14}$$

In particular, at $n=0$ we obtain differential equation for associated Legendre functions:

$$\left[(1-x^2)\frac{d^2}{dx^2}-2x\frac{d}{dx}-\frac{m^2}{1-x^2}\right]P^m_l(x)=-l(l+1)P^m_l(x).$$

Further, at $n=0$ and $m=0$ we come to the well known differential equation for Legendre polynomials:

$$(1-x^2)\frac{d^2P_l(x)}{dx^2}-2x\frac{dP_l(x)}{dx}+l(l+1)P_l(x)=0.$$

2.1 Harmonic analysis on the group $SU(2)$

It is well known [90, 120] that the classical Fourier analysis is completely formulated in terms of an additive group of real numbers, $\mathbb{R}$. $\mathbb{R}$ is an Abelian and non-compact group, and by this reason all its unitary representations are one-dimensional and expressed via the exponential e^{ax}, where $a=bi$. The regular representation of the group $\mathbb{R}$ is constructed in the space $\mathfrak{S}$ of the square integrable functions $f(x)$ defined on the group $\mathbb{R}$ (equally, on the real axis $-\infty<x<\infty$) such that

$$\int\limits_{-\infty}^{\infty}|f(x)|^2dx<+\infty.$$

The operator $\mathbb{R}(x_0)$, transforming the function $f(x)$ into $\mathbb{R}(x_0)f(x)=f(x+x_0)$, corresponds to the each element of the group $\mathbb{R}$. $\mathbb{R}(x)$ is a regular representation of $\mathbb{R}$. At this point, any function $f(x)\in\mathfrak{S}$ can be represented in the form of continuous linear combination of the functions $e^{i\lambda x}$ (Fourier integral):

$$f(x)=\int\limits_{-\infty}^{\infty}F(\lambda)e^{i\lambda x}dx,\quad -\infty<x<\infty.$$

The inverse transformation is defined by a formula

$$F(\lambda)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}f(x)e^{-i\lambda x}dx.$$

Hence it follows a so called Plancherel formula

$$\int\limits_{-\infty}^{\infty}|F(\lambda)|^2d\lambda=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}|f(x)|^2dx.$$

In the case of a quotient group $SO(2)=\mathbb{R}/\mathbb{Z}_{2\pi}$ (rotation group of the euclidean plane), where the subgroup $\mathbb{Z}_{2\pi}$ is generated by numbers of the form $2\pi k$, we come to the classical

Fourier series. Indeed, if $f(\varphi)$ is an irreducible unitary representation of $SO(2)$, then the equality

$$F(\varphi + 2\pi k) = f(\varphi), \quad 0 \le \varphi < 2\pi,$$

defines the irreducible unitary representation e^{iax} of $\mathbb{R}$. The representations of $SO(2)$ are derived from e^{iax} at the conditions $F(2\pi) = f(0) = 1$. They have the form $f(\varphi) = e^{in\varphi}$, where n is an integer number.

The integral on the group $SO(2)$ is defined as follows

$$\int f(g)dg = \frac{1}{2\pi}\int\limits_{0}^{2\pi} f(\varphi)d\varphi,$$

where $1/2\pi d\varphi$ is an invariant measure (Haar measure) on the group $SO(2)$. This integral possesses the property

$$\int f(gg_0)dg = \int f(g)dg.$$

Any square integrable function $f(\varphi)$ on the group $SO(2)$ expands into a convergent series

$$f(\varphi) = \sum_{n=-\infty}^{\infty} c_n e^{in\varphi},$$

where the coefficients are expressed via the formulae

$$c_n = \frac{1}{2\pi}\int\limits_{0}^{2\pi} f(\varphi)e^{-in\varphi}d\varphi.$$

Hence it immediately follows a Parseval equality

$$\frac{1}{2\pi}\int\limits_{0}^{2\pi} |f(\varphi)|^2 d\varphi = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

The generalization of classical Fourier series for the case of non-Abelian compact groups was first given by Peter and Weyl [83]. The first simplest case of such groups is $SU(2)$. Since $SU(2)$ is compact, then there exists an invariant measure du on this group such that

$$\int f(u)du = \int f(u_0 u)du = \int f(uu_0)du = \int f(u^{-1})du. \tag{15}$$

This equality holds for all continuous functions $f(u)$ and all elements u_0 from $SU(2)$. The invariant integral on the group $SU(2)$ is defined in Euler parameterization by a formula

$$\int\limits_{SU(2)} f(u)du = \frac{1}{16\pi^2}\int\limits_{-2\pi}^{2\pi}\int\limits_{0}^{2\pi}\int\limits_{0}^{\pi} f(\varphi,\theta,\psi)\sin\theta d\theta d\varphi d\psi, \tag{16}$$

where $1/16\pi^2 \sin\theta d\theta d\varphi d\psi$ is a Haar measure on the group $SU(2)$. It is easy to verify that the integral (16) satisfy the equality (15).

Since the dimensionality of the representation $T_l(u)$ of $SU(2)$ is equal to $2l+1$, then the functions $\sqrt{2l+1}t^l_{mn}(u)$ form a full orthogonal normalized system of functions with respect to the invariant measure du on this group. In other words, the functions $t^l_{mn}(u)$ (matrix elements) satisfy the relations

$$\int\limits_{SU(2)} t^l_{mn}(u)\overline{t^s_{pq}(u)}du = \frac{1}{2l+1}\delta_{ls}\delta_{mp}\delta_{nq}. \tag{17}$$

Substituting into (17) instead the matrix elements $t^l_{mn}(u)$ their expressions via the Euler angles (see (12)), we come to the following formula

$$\int\limits_{-2\pi}^{2\pi}\int\limits_{0}^{2\pi}\int\limits_{0}^{\pi} P^l_{mn}(\cos\theta)\overline{P^s_{pq}(\cos\theta)}\sin\theta e^{i(p-m)\varphi}e^{i(q-n)\psi}d\theta d\varphi d\psi = \frac{16\pi^2}{2l+1}\delta_{ls}\delta_{mp}\delta_{nq}. \tag{18}$$

Due to the well known orthogonality relations for the functions $P^l_{mn}(x)$ [120],

$$\int\limits_{-1}^{1} P^l_{mn}(x)\overline{P^s_{mn}(x)}dx = \frac{2}{2l+1}\delta_{ls},$$

we see that the relation (18) holds for all l, s, m, p, n, q.

Thus, any square integrable function $f(\varphi,\theta,\psi)$ on the group $SU(2)$, $0\le\varphi<2\pi$, $0\le\theta<\pi$, $-2\pi<\psi<2\pi$, such that

$$\int\limits_{-2\pi}^{2\pi}\int\limits_{0}^{2\pi}\int\limits_{0}^{\pi} |f(\varphi,\theta,\psi)|^2\sin\theta d\theta d\varphi d\psi \;<\; +\infty,$$

is expanded in a convergent Fourier series on $SU(2)$,

$$f(\varphi,\theta,\psi) = \sum_l\sum_{m=-l}^{l}\sum_{n=-l}^{l}\alpha^l_{mn}e^{-i(m\varphi+n\psi)}P^l_{mn}(\cos\theta),$$

where

$$\alpha^l_{mn} = \frac{(-1)^{m-n}(2l+1)}{16\pi^2}\int\limits_{-2\pi}^{2\pi}\int\limits_{0}^{2\pi}\int\limits_{0}^{\pi} f(\varphi,\theta,\psi)e^{i(m\varphi+n\psi)}P^l_{mn}(\cos\theta)\sin\theta d\theta d\varphi d\psi.$$

The Parseval equality in this case has a form

$$\sum_l\sum_{m=-l}^{l}\sum_{n=-l}^{l}|\alpha^l_{mn}|^2 = \frac{2l+1}{16\pi^2}\int\limits_{-2\pi}^{2\pi}\int\limits_{0}^{2\pi}\int\limits_{0}^{\pi} |f(\varphi,\theta,\psi)|^2\sin\theta d\theta d\varphi d\psi.$$

3 The group $SU(1,1)$

The group $SU(1,1)$ (also known as a three-dimensional Lorentz group) is an universal covering of the group $SH(3)$. The group $SH(3)$ is a group of linear transformations of the space E_3 with a quadratic form

$$[\mathbf{x},\mathbf{x}] = x_1^2 - x_2^2 - x_3^2.$$

Any matrix $g \in SU(1,1)$ has the form

$$g = \begin{pmatrix} \gamma & \delta \\ \bar{\delta} & \bar{\gamma} \end{pmatrix}, \tag{19}$$

where $\det g = 1$ and, therefore, $|\gamma|^2 - |\delta|^2 = 1$. An Euler parametrization of (19) is

$$g = \begin{pmatrix} \cosh\frac{\tau}{2}e^{\frac{i(\varphi+\psi)}{2}} & \sinh\frac{\tau}{2}e^{\frac{i(\varphi-\psi)}{2}} \\ \sinh\frac{\tau}{2}e^{\frac{i(\psi-\varphi)}{2}} & \cosh\frac{\tau}{2}e^{\frac{-i(\varphi+\psi)}{2}} \end{pmatrix},$$

where $0 \le \varphi \le 2\pi$, $0 < \tau < \infty$, $-2\pi \le \psi < 2\pi$. Hence it immediately follows a Cartan decomposition for the group $SU(1,1)$:

$$g(\varphi,\tau,\psi) = \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix}\begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{pmatrix}\begin{pmatrix} e^{\frac{i\psi}{2}} & 0 \\ 0 & e^{-\frac{i\psi}{2}} \end{pmatrix}. \tag{20}$$

The group $SU(1,1)$ has three one-parameter subgroups

$$\omega_1(t) = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \quad \omega_2(t) = \begin{pmatrix} \cosh\frac{t}{2} & i\sinh\frac{t}{2} \\ -i\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \quad \omega_3(t) = \begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix}.$$

The tangent matrices A_i of these subgroups have the form

$$\begin{aligned}
A_1 &= \left.\frac{d\omega_1(t)}{dt}\right|_{t=0} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
A_2 &= \left.\frac{d\omega_2(t)}{dt}\right|_{t=0} = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
A_3 &= \left.\frac{d\omega_3(t)}{dt}\right|_{t=0} = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}$$

The elements A_i form a basis of Lie algebra $\mathfrak{su}(1,1)$ and satisfy the relations

$$[A_1,A_2] = -A_3, \quad [A_2,A_3] = A_1, \quad [A_3,A_1] = A_2.$$

In this case an Euler parametrization of infinitesimal operators A_i for the left regular representation has a form

$$\mathsf{A}_1 = \coth\tau\sin\varphi\frac{\partial}{\partial\varphi} - \cos\varphi\frac{\partial}{\partial\tau} - \frac{\sin\varphi}{\sinh\tau}\frac{\partial}{\partial\psi}, \tag{21}$$

$$\mathsf{A}_2 = -\coth\tau\cos\varphi\frac{\partial}{\partial\varphi} - \sin\varphi\frac{\partial}{\partial\tau} + \frac{\cos\varphi}{\sinh\tau}\frac{\partial}{\partial\psi}, \tag{22}$$

$$\mathsf{A}_3 = -\frac{\partial}{\partial\varphi}. \tag{23}$$

Since $SU(1,1) \simeq SL(2,\mathbb{R})$ is the noncompact group, then in this case we have both finite- and infinite-dimensional representations. Representations of the group $SL(2,\mathbb{R})$ are defined by the two numbers $\chi = (\tau, o)$, $\tau \in \mathbb{C}$, $o \in \{0, 1/2\}$ [120]. The representation $T_\chi \equiv T_{(\tau,o)}$ of the principal nonunitary series of $SL(2,\mathbb{R})$ is realized in the space of functions $f(x)$ which depend on the real variable:

$$T_\chi(g)f(x) = |\beta x + \delta|^{2\tau}\operatorname{sign}^{2o}(\beta x + \delta) f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right),$$

where $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R})$. At $\tau = i\rho - \frac{1}{2}$, $\rho \in \mathbb{R}$, the representations T_χ are unitary with respect to a scalar product in the Hilbert space $L^2(\mathbb{R})$. They form the principal unitary representation series of $SL(2,\mathbb{R})$. The group $SL(2,\mathbb{R})$ has also discrete series of unitary representations. Representations of the negative discrete series T_l^-, $l = -1, -\frac{3}{2}, -2, -\frac{5}{2}, \ldots$, act in the Hilbert space H_l of the functions $F(w)$ which are analytic in upper half-plane $\mathbb{C}_+$ and possess with a following scalar product

$$(F_1, F_2) = \frac{i}{2\Gamma(-2l-1)} \int\limits_{\mathbb{C}_+} F_1(w)\overline{F_2(w)} y^{-2l-2} dw d\overline{w},$$

where $w = x + iy$, $dwd\overline{w} = -2idxdy$. The operators $T_l^-(g)$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, are defined by

$$T_l^-(g)F(w) = (\beta w + \delta)^{2l} F\left(\frac{\alpha w + \gamma}{\beta w + \delta}\right).$$

In like manner, representations of the positive discrete series T_l^+, $l = -1, -\frac{3}{2}, -2, \ldots$, are constructed in the Hilbert space of functions which are analytic in lower half-plane $\mathbb{C}_-$. Since, $SU(1,1) \simeq SL(2,\mathbb{R}$, then representations of the principal nonunitary series of $SU(1,1)$ are defined by the same pair $\chi = (l, o)$ as with the group $SL(2,\mathbb{R})$. These representations are realized on the subgroup $K = SO(2)$:

$$T_\chi(g)f(e^{i\theta}) = (\gamma e^{i\theta} + \bar{\delta})^{l+o}(\bar{\gamma} e^{-i\theta} + \delta)^{l-o} f\left(\frac{\delta e^{i\theta} + \bar{\gamma}}{\gamma e^{i\theta} + \bar{\delta}}\right), \tag{24}$$

where $g = \begin{pmatrix} \gamma & \delta \\ \bar{\delta} & \bar{\gamma} \end{pmatrix} \in SU(1,1)$.

As follows from (24), matrix elements $t^\chi_{mn}(g)$ of the representation $T_\chi(g)$ in the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{ip\varphi} \,\middle|\, p = 0, \pm 1, \pm 2, \ldots \right\}$$

have a following integral representation

$$t^\chi_{mn}(g) = \frac{1}{2\pi} \int\limits_0^{2\pi} (\gamma e^{i\theta} + \bar{\delta})^{l+n+o} (\bar{\gamma} e^{-i\theta} + \delta)^{l-n-o} e^{i(m-n)\varphi} d\varphi,$$

where m and n are integer or half-integer numbers in accordance with the values of o. This integral can be calculated by means of the Newton binomial. In the case of finite-dimensional

representations we have

$$t^{\chi}_{mn}(g)=\Gamma(l+n+o+1)\Gamma(l-n-o+1)\delta^{l-n-o}\overline{\delta}^{l+m+o}\gamma^{n-m}\times$$
$$\sum_{s=\max(0,m-n)}^{\min(l-n,l+m)}\frac{|\gamma/\delta|^{2s}}{\Gamma(s+1)\Gamma(l-n-s-o+1)\Gamma(n-m+s+1)\Gamma(l+m-s+o+1)}. \quad (25)$$

In accordance with the Cartan decomposition (20), the matrix elements can be written as follows

$$\begin{aligned} t^{\chi}_{mn}(g) &= t^{\chi}_{m'm'}[g(\varphi,0,0)]t^{\chi}_{m'n'}(\tau)t^{\chi}_{n'n'}[g(0,0,\psi)] \\ &= e^{-i(m'\varphi+n'\psi)}\mathfrak{P}^{l}_{m'n'}(\cosh\tau), \end{aligned} \quad (26)$$

where $\chi=(l,o)$, $m'=m+o$, $n'=n+o$. An explicit form of the function $\mathfrak{P}^{l}_{mn}(\cosh\tau)$ (Jacobi function) follows from (25) at $\gamma=\sinh\frac{\tau}{2}$ and $\delta=\cosh\frac{\tau}{2}$:

$$\mathfrak{P}^{l}_{mn}(\cosh\tau)=\sqrt{\Gamma(l-n+1)\Gamma(l+n+1)\Gamma(l-m+1)\Gamma(l+m+1)}\times$$
$$\cosh^{2l}\frac{\tau}{2}\tanh^{n-m}\frac{\tau}{2}\times$$
$$\sum_{s=\max(0,m-n)}^{\min(l-n,l+m)}\frac{\tanh^{2s}\frac{\tau}{2}}{\Gamma(s+1)\Gamma(l-n-s+1)\Gamma(n-m+s+1)\Gamma(l+m-s+1)}, \quad (27)$$

where $m,n=-l,-l+1,\dots,l$.

Let us give explicit expressions for the matrices $T_l(\tau)$ at $l=0$, $1/2$, 1:

$$T_0(\tau)=1,$$
$$T_{\frac{1}{2}}(\tau)=\begin{pmatrix}\mathfrak{P}^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}} & \mathfrak{P}^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}}\\ \mathfrak{P}^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}} & \mathfrak{P}^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}\end{pmatrix}=\begin{pmatrix}\cosh\frac{\tau}{2} & \sinh\frac{\tau}{2}\\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2}\end{pmatrix},$$
$$T_1(\tau)=\begin{pmatrix}\mathfrak{P}^{1}_{-1-1} & \mathfrak{P}^{1}_{-10} & \mathfrak{P}^{1}_{-11}\\ \mathfrak{P}^{1}_{0-1} & \mathfrak{P}^{1}_{00} & \mathfrak{P}^{1}_{01}\\ \mathfrak{P}^{1}_{1-1} & \mathfrak{P}^{1}_{10} & \mathfrak{P}^{1}_{11}\end{pmatrix}=\begin{pmatrix}\cosh^2\frac{\tau}{2} & \sinh\tau & \sinh^2\frac{\tau}{2}\\ \sinh\tau & \cosh\tau & \sinh\tau\\ \sinh^2\frac{\tau}{2} & \sinh\tau & \cosh^2\frac{\tau}{2}\end{pmatrix}.$$

Using the formula (9), we can express the function $\mathfrak{P}^{l}_{mn}(\cosh\tau)$ via the hypergeometric function:

$$\mathfrak{P}^{l}_{mn}(\cosh\tau)=\frac{1}{\Gamma(n-m+1)}\sqrt{\frac{\Gamma(l+n+1)\Gamma(l-m+1)}{\Gamma(l-n+1)\Gamma(l+m+1)}}\times$$
$$\cosh^{2l}\frac{\tau}{2}\tanh^{n-m}\frac{\tau}{2}\,{}_2F_1\left(\begin{array}{c}n-l+1,1-l-m\\ n-m+1\end{array}\middle|\tanh^2\frac{\tau}{2}\right), \quad (28)$$

where $m\geq n$. At $m<n$ in the right part of (28) it needs to replace m and n by $-m$ and $-n$, respectively. Since l, m and n are finite numbers, then the hypergeometric series in (28) is finite also.

In the case of principal series of unitary representations, the function (27) takes a form

$$\mathfrak{P}^{-\frac{1}{2}+i\rho}_{mn}(\cosh\tau)=\sqrt{\Gamma(i\rho-n+\tfrac{1}{2})\Gamma(i\rho+n+\tfrac{1}{2})\Gamma(i\rho-m+\tfrac{1}{2})\Gamma(i\rho+m+\tfrac{1}{2})}\times$$
$$\cosh^{2i\rho-1}\frac{\tau}{2}\tanh^{n-m}\frac{\tau}{2}\times$$
$$\sum^{\infty}_{s=\max(0,m-n)}\frac{\tanh^{2s}\frac{\tau}{2}}{\Gamma(s+1)\Gamma(i\rho-n-s+\frac{1}{2})\Gamma(n-m+s+1)\Gamma(i\rho+m-s+\frac{1}{2})},$$

or

$$\mathfrak{P}^{-\frac{1}{2}+i\rho}_{mn}(\cosh\tau)=\frac{1}{\Gamma(n-m+1)}\sqrt{\frac{\Gamma(i\rho+n+\frac{1}{2})\Gamma(i\rho-m+\frac{1}{2})}{\Gamma(i\rho-n+\frac{1}{2})\Gamma(i\rho+m+\frac{1}{2})}}\times$$
$$\cosh^{2i\rho-1}\frac{\tau}{2}\tanh^{n-m}\frac{\tau}{2}{}_2F_1\left(\begin{matrix}n-i\rho+\frac{3}{2},\frac{3}{2}-i\rho-m\\ n-m+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right). \quad (29)$$

When $m=0$, $n=0$ (zonal functions), the function (29) transforms into *a conical function* $\mathfrak{P}_{i\rho-\frac{1}{2}}(\cosh\tau)$ (for more details about conical functions see [9]).

Matrix elements $t^{l,-}_{mn}(g_\tau)$, $-\infty<m,n\leq l$, of the discrete series T^-_l of representations of the group $SU(1,1)$ are expressed via the Jacobi polynomials $P^{(\alpha,\beta)}_n(\cosh\tau)$ [121]:

$$\mathfrak{P}^{l,-}_{mn}(\cosh\tau)=\left[\frac{\Gamma(l-m+1)\Gamma(-l-n)}{\Gamma(l-n+1)\Gamma(-l-m)}\right]^{\frac{1}{2}}\sinh^{m-n}\frac{\tau}{2}\cosh^{m+n}\frac{\tau}{2}P^{(m-n,m+n)}_{l-m}(\cosh\tau)$$

if $n\leq m\leq l$ and

$$\mathfrak{P}^{l,-}_{mn}(\cosh\tau)=\left[\frac{\Gamma(l-n+1)\Gamma(-l-m)}{\Gamma(l-m+1)\Gamma(-l-n)}\right]^{\frac{1}{2}}\sinh^{n-m}\frac{\tau}{2}\cosh^{m+n}\frac{\tau}{2}P^{(n-m,m+n)}_{l-n}(\cosh\tau)$$

if $m\leq n\leq l$.

On the group $SU(1,1)$ there exists the following Laplace-Beltrami operator (or Casimir operator):

$$\Delta=-\mathsf{A}^2_1-\mathsf{A}^2_2+\mathsf{A}^2_3.$$

Using (21)–(22), we can express this operator via the Euler angles:

$$\Delta=-\frac{1}{\sinh\tau}\frac{\partial}{\partial\tau}\sinh\tau\frac{\partial}{\partial\tau}-\frac{1}{\sinh^2\tau}\left[\frac{\partial^2}{\partial\varphi^2}-2\cosh\tau\frac{\partial^2}{\partial\varphi\partial\psi}+\frac{\partial^2}{\partial\psi^2}\right]. \quad (30)$$

Matrix elements $t^\chi_{mn}(g)$ of irreducible representations of the group $SU(1,1)$ are eigenfunctions of the operator (30):

$$\Delta t^\chi_{mn}(g)=l(l+1)t^\chi_{mn}(g).$$

Indeed, substituting in this equality the expression (30) and Euler parameterization of $t^\chi_{mn}(g)$ instead Δ and $t^\chi_{mn}(g)$ and supposing $y=\cosh\tau$, we come to a following differential equation

$$\left[(y^2-1)\frac{d^2}{dy^2}+2y\frac{d}{dy}-\frac{m^2+n^2-2mny}{y^2-1}\right]\mathfrak{P}^l_{mn}(y)=l(l+1)\mathfrak{P}^l_{mn}(y).$$

It is easy to see that this equation has the structure similar to the equation (14) for the functions $P^l_{mn}(x)$.

3.1 Harmonic analysis on the group $SU(1,1)$

Harmonic analysis on the group $SU(1,1)$ was studied by many authors during long time (see, for example, [61, 87, 120, 66]). Let $C_0^\infty(G)$, $G = SL(2,\mathbb{R})$, be a space of infinitely differentiable functions on the group $SL(2,\mathbb{R})$, then for the representation T_χ of the principal nonunitary series of $SL(2,\mathbb{R})$ there exists on $C_0^\infty(G)$ the following transformation (Fourier transform)

$$T_\chi^f = \int\limits_{SL(2,\mathbb{R})} f(g) T_\chi^*(g) dg, \quad f \in C_0^\infty(G). \tag{31}$$

The same transformation can be defined for the representations T_l^- and T_l^+ of the discrete series of $SL(2,\mathbb{R})$:

$$T_l^{f,\pm} = \int\limits_{SL(2,\mathbb{R})} f(g) \left[T_l^\pm(g)\right]^* dg, \tag{32}$$

where T_χ^f, $T_\chi^{f,\pm}$ are operators with the trace.

The inverse Fourier transform is defined by a formula

$$f(g) = \sum_{o=0,\frac{1}{2}} \frac{1}{4\pi^2} \int\limits_0^\infty \operatorname{Tr}\left[T^f_{(i\rho-1/2,o)} T_{(i\rho-1/2,o)}(g)\right] \rho \tanh \pi(\rho + io) dp +$$
$$+ \frac{1}{4\pi^2} \sum_l \left(l + \frac{1}{2}\right) \operatorname{Tr}\left[T^{f,+}_{-l-1} T^+_{-l-1}(g) + T^{f,-}_{-l-1} T^-_{-l-1}(g)\right], \tag{33}$$

where the index l in the second sum runs the values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

In this case the Plancherel formula can be written as follows

$$\int\limits_{SL(2,\mathbb{R})} |f(g)|^2 dg = \frac{1}{4\pi^2} \left\{ \sum_{o=0,\frac{1}{2}} \int\limits_0^\infty \operatorname{Tr}\left[T^f_{(i\rho-1/2,o)} (T^f_{(i\rho-1/2,o)})^*\right] \times \right.$$
$$\times \rho \tanh \pi(\rho + io) d\rho + \sum_l \left(l + \frac{1}{2}\right) \operatorname{Tr}\left[T^{f,+}_{-l-1} (T^{f,+}_{-l-1})^* + \right.$$
$$\left.\left. + T^{f,-}_{-l-1} (T^{f,-}_{-l-1})^*\right]\right\}. \tag{34}$$

The invariant measure on the group $SU(1,1)$ has a form

$$dg = \frac{1}{4\pi^2} \sinh \tau d\varphi d\tau d\psi.$$

Taking into account this expression, we can rewrite the formulae (31)–(34) via the matrix elements (25). Indeed, any square integrable function $f(g)$ on the group $SU(1,1)$, such that

$$\int\limits_{SU(1,1)} |f(g)|^2 dg < +\infty,$$

is expanded in matrix elements as follows

$$f(\varphi,\tau,\psi)=\frac{1}{4\pi^2}\sum_{m,n,o}\left[\int\limits_0^\infty a^o_{mn}(\rho)e^{-i(m\varphi+n\psi)}\mathfrak{P}^{-\frac{1}{2}+i\rho,o}_{mn}(\cosh\tau)\rho\tanh\pi(\rho+oi)d\rho+\right.$$
$$\left.+\sum_{l=1-o}^{\infty}\left(l-\frac{1}{2}\right)b^o_{mn}(l)e^{-i(m\varphi+n\psi)}\mathfrak{P}^{l,o}_{mn}(\cosh\tau)\right],\quad (35)$$

where the values of the coefficients $a^o_{mn}(\rho)$ and $b^o_{mn}(l)$ are expressed via the formulae

$$a^o_{mn}(\rho)=\int\limits_{-2\pi}^{2\pi}\int\limits_0^{2\pi}\int\limits_0^\infty f(\varphi,\tau,\psi)e^{i(m\varphi+n\psi)}\overline{\mathfrak{P}^{-\frac{1}{2}+i\rho,o}_{mn}(\cosh\tau)}\sinh\tau d\tau d\varphi d\psi,$$

$$b^o_{mn}(l)=\frac{(-1)^{m-n}\Gamma(l+m+o+1)\Gamma(l-m-o+1)}{\Gamma(l+n+o+1)\Gamma(l-n-o+1)}\times$$
$$\int\limits_{-2\pi}^{2\pi}\int\limits_0^{2\pi}\int\limits_0^\infty f(\varphi,\tau,\psi)e^{i(m\varphi+n\psi)}\overline{\mathfrak{P}^{l,o}_{mn}(\cosh\tau)}\sinh\tau d\tau d\varphi d\psi.$$

Further, the Plancherel formula takes a form

$$\int\limits_{SU(1,1)}|f(g)|^2dg=\frac{1}{4\pi^2}\sum_{m,n,o}\left[\int\limits_0^\infty|a^o_{mn}(\rho)|^2\rho\tanh\pi(\rho+oi)d\rho+\right.$$
$$\left.+(-1)^{m-n}\sum_{l=1-o}^{\infty}\frac{\Gamma(l+n+o+1)\Gamma(l-n-o+1)}{\Gamma(l+m+o+1)\Gamma(l-m-o+1)}\left(l-\frac{1}{2}\right)|b^o_{mn}(l)|^2\right].$$

4 The group $SL(2,\mathbb{C})$

As is known, the group $SL(2,\mathbb{C})$ is an universal covering of the proper orthochronous Lorentz group $L^\uparrow_+$. The group $SL(2,\mathbb{C})$ of all complex matrices

$$\mathfrak{g}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

of 2-nd order with the determinant $\alpha\delta-\gamma\beta=1$, is a *complexification* of the group $SU(2)$. The group $SU(2)$ is one of the real forms of $SL(2,\mathbb{C})$. The transition from $SU(2)$ to $SL(2,\mathbb{C})$ is realized via the complexification of three real parameters φ, θ, ψ (Euler angles). Let $\theta^c=\theta-i\tau$, $\varphi^c=\varphi-i\epsilon$, $\psi^c=\psi-i\varepsilon$ be complex Euler angles, where

$$\begin{array}{rcccclrcccl}
0 & \leq & \operatorname{Re}\theta^c=\theta & \leq & \pi, & & -\infty & < & \operatorname{Im}\theta^c=\tau & < & +\infty,\\
0 & \leq & \operatorname{Re}\varphi^c=\varphi & < & 2\pi, & & -\infty & < & \operatorname{Im}\varphi^c=\epsilon & < & +\infty,\\
-2\pi & \leq & \operatorname{Re}\psi^c=\psi & < & 2\pi, & & -\infty & < & \operatorname{Im}\psi^c=\varepsilon & < & +\infty.
\end{array}\quad (36)$$

Replacing in (2) the angles φ, θ, ψ by the complex angles $\varphi^c, \theta^c, \psi^c$, we come to the following matrix

$$\mathfrak{g} = \begin{pmatrix} \cos\frac{\theta^c}{2}e^{\frac{i(\varphi^c+\psi^c)}{2}} & i\sin\frac{\theta^c}{2}e^{\frac{i(\varphi^c-\psi^c)}{2}} \\ i\sin\frac{\theta^c}{2}e^{\frac{i(\psi^c-\varphi^c)}{2}} & \cos\frac{\theta^c}{2}e^{-\frac{i(\varphi^c+\psi^c)}{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \left[\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right]e^{\frac{\epsilon+\varepsilon+i(\varphi+\psi)}{2}} & \left[\cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right]e^{\frac{\epsilon-\varepsilon+i(\varphi-\psi)}{2}} \\ \left[\cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right]e^{\frac{\varepsilon-\epsilon+i(\psi-\varphi)}{2}} & \left[\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right]e^{\frac{-\epsilon-\varepsilon-i(\varphi+\psi)}{2}} \end{pmatrix}, \quad (37)$$

since $\cos\frac{1}{2}(\theta-i\tau) = \cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}$, and $\sin\frac{1}{2}(\theta-i\tau) = \sin\frac{\theta}{2}\cosh\frac{\tau}{2}-i\cos\frac{\theta}{2}\sinh\frac{\tau}{2}$. It is easy to see that in this case a Cartan decomposition for $SL(2,\mathbb{C})$ has the form

$$\mathfrak{g}(\varphi,\,\epsilon,\,\theta,\,\tau,\,\psi,\,\varepsilon) =$$

$$\begin{pmatrix} e^{i\frac{\varphi}{2}} & 0 \\ 0 & e^{-i\frac{\varphi}{2}} \end{pmatrix}\begin{pmatrix} e^{\frac{\epsilon}{2}} & 0 \\ 0 & e^{-\frac{\epsilon}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{pmatrix}\begin{pmatrix} e^{i\frac{\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix}\begin{pmatrix} e^{\frac{\varepsilon}{2}} & 0 \\ 0 & e^{-\frac{\varepsilon}{2}} \end{pmatrix}. \quad (38)$$

If we restrict the parameters $\operatorname{Im}\varphi^c = \epsilon$, $\operatorname{Im}\psi^c = \varepsilon$ within the limits $0 \le \epsilon \le 2\pi$, $-2\pi \le \varepsilon < 2\pi$, then we come to the following Cartan decomposition

$$\mathfrak{g} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} \gamma & \delta \\ \bar{\delta} & \bar{\gamma} \end{pmatrix}, \quad (39)$$

where $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)$ and $\begin{pmatrix} \gamma & \delta \\ \bar{\delta} & \bar{\gamma} \end{pmatrix} \in SU(1,1)$.

The group $SL(2,\mathbb{C})$ has six one-parameter subgroups

$$a_1(t) = \begin{pmatrix} \cos\frac{t}{2} & i\sin\frac{t}{2} \\ i\sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}, \quad a_2(t) = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}, \quad a_3(t) = \begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix},$$

$$b_1(t) = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \quad b_2(t) = \begin{pmatrix} \cosh\frac{t}{2} & i\sinh\frac{t}{2} \\ -i\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \quad b_3(t) = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}.$$

The tangent matrices A_i and B_i of these subgroups are defined as follows

$$\begin{aligned}
A_1 &= \left.\frac{da_1(t)}{dt}\right|_{t=0} = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
A_2 &= \left.\frac{da_2(t)}{dt}\right|_{t=0} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
A_3 &= \left.\frac{da_3(t)}{dt}\right|_{t=0} = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
B_1 &= \left.\frac{db_1(t)}{dt}\right|_{t=0} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
B_2 &= \left.\frac{db_2(t)}{dt}\right|_{t=0} = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
B_3 &= \left.\frac{db_3(t)}{dt}\right|_{t=0} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}$$

The elements A_i and B_i form a basis of Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and satisfy the relations

$$\left.\begin{array}{lll}
[\mathsf{A}_1,\mathsf{A}_2]=\mathsf{A}_3, & [\mathsf{A}_2,\mathsf{A}_3]=\mathsf{A}_1, & [\mathsf{A}_3,\mathsf{A}_1]=\mathsf{A}_2,\\
[\mathsf{B}_1,\mathsf{B}_2]=-\mathsf{A}_3, & [\mathsf{B}_2,\mathsf{B}_3]=-\mathsf{A}_1, & [\mathsf{B}_3,\mathsf{B}_1]=-\mathsf{A}_2,\\
[\mathsf{A}_1,\mathsf{B}_1]=0, & [\mathsf{A}_2,\mathsf{B}_2]=0, & [\mathsf{A}_3,\mathsf{B}_3]=0,\\
[\mathsf{A}_1,\mathsf{B}_2]=\mathsf{B}_3, & [\mathsf{A}_1,\mathsf{B}_3]=-\mathsf{B}_2, & \\
[\mathsf{A}_2,\mathsf{B}_3]=\mathsf{B}_1, & [\mathsf{A}_2,\mathsf{B}_1]=-\mathsf{B}_3, & \\
[\mathsf{A}_3,\mathsf{B}_1]=\mathsf{B}_2, & [\mathsf{A}_3,\mathsf{B}_2]=-\mathsf{B}_1. &
\end{array}\right\} \qquad (40)$$

Denoting $\mathsf{I}^{23}=\mathsf{A}_1$, $\mathsf{I}^{31}=\mathsf{A}_2$, $\mathsf{I}^{12}=\mathsf{A}_3$, and $\mathsf{I}^{01}=\mathsf{B}_1$, $\mathsf{I}^{02}=\mathsf{B}_2$, $\mathsf{I}^{03}=\mathsf{B}_3$ we can write the relations (40) in a more compact form:

$$\left[\mathsf{I}^{\mu\nu},\mathsf{I}^{\lambda\rho}\right]=\delta_{\mu\rho}\mathsf{I}^{\lambda\nu}+\delta_{\nu\lambda}\mathsf{I}^{\mu\rho}-\delta_{\nu\rho}\mathsf{I}^{\mu\lambda}-\delta_{\mu\lambda}\mathsf{I}^{\nu\rho}.$$

Let us consider the operators

$$\mathsf{X}_l=\frac{1}{2}i(\mathsf{A}_l+i\mathsf{B}_l),\quad \mathsf{Y}_l=\frac{1}{2}i(\mathsf{A}_l-i\mathsf{B}_l), \qquad (41)$$
$$(l=1,2,3).$$

Using the relations (40), we find that

$$[\mathsf{X}_k,\mathsf{X}_l]=i\varepsilon_{klm}\mathsf{X}_m,\quad [\mathsf{Y}_l,\mathsf{Y}_m]=i\varepsilon_{lmn}\mathsf{Y}_n,\quad [\mathsf{X}_l,\mathsf{Y}_m]=0. \qquad (42)$$

Further, introducing generators of the form

$$\left.\begin{array}{ll}
\mathsf{X}_+=\mathsf{X}_1+i\mathsf{X}_2, & \mathsf{X}_-=\mathsf{X}_1-i\mathsf{X}_2,\\
\mathsf{Y}_+=\mathsf{Y}_1+i\mathsf{Y}_2, & \mathsf{Y}_-=\mathsf{Y}_1-i\mathsf{Y}_2,
\end{array}\right\} \qquad (43)$$

we see that in virtue of commutativity of the relations (42) a space of an irreducible finite–dimensional representation of the group $SL(2,\mathbb{C})$ can be spanned on the totality of $(2l+1)(2\dot{l}+1)$ basis vectors $\mid l,m;\dot{l},\dot{m}\rangle$, where $l,m,\dot{l},\dot{m}$ are integer or half–integer numbers, $-l\le m\le l$, $-\dot{l}\le\dot{m}\le\dot{l}$. Therefore,

$$\begin{array}{l}
\mathsf{X}_-\mid l,m;\dot{l},\dot{m}\rangle=\sqrt{(l+m)(l-m+1)}\mid l,m-1,\dot{l},\dot{m}\rangle\ \ (m>-l),\\
\mathsf{X}_+\mid l,m;\dot{l},\dot{m}\rangle=\sqrt{(l-m)(l+m+1)}\mid l,m+1;\dot{l},\dot{m}\rangle\ \ (m<l),\\
\mathsf{X}_3\mid l,m;\dot{l},\dot{m}\rangle=m\mid l,m;\dot{l},\dot{m}\rangle,\\
\mathsf{Y}_-\mid l,m;\dot{l},\dot{m}\rangle=\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}+1)}\mid l,m;\dot{l},\dot{m}-1\rangle\ \ (\dot{m}>-\dot{l}),\\
\mathsf{Y}_+\mid l,m;\dot{l},\dot{m}\rangle=\sqrt{(\dot{l}-\dot{m})(\dot{l}+\dot{m}+1)}\mid l,m;\dot{l},\dot{m}+1\rangle\ \ (\dot{m}<\dot{l}),\\
\mathsf{Y}_3\mid l,m;\dot{l},\dot{m}\rangle=\dot{m}\mid l,m;\dot{l},\dot{m}\rangle.
\end{array} \qquad (44)$$

From the relations (42) it follows that each of the sets of infinitesimal operators X and Y generates the group $SU(2)$ and these two groups commute with each other. Thus, from the relations (42) and (44) it follows that the group $SL(2,\mathbb{C})$, in essence, is equivalent locally to the group $SU(2)\otimes SU(2)$. In contrast to the Gel'fand–Naimark representation for the Lorentz group [36, 78], which does not find a broad application in physics, a representation (44) is a most useful in theoretical physics (see, for example, [1, 95, 92, 93]). This representation for

the Lorentz group was first given by Van der Waerden in his brilliant book [122]. It should be noted here that the representation basis, defined by the formulae (41)–(44), has an evident physical meaning. For example, in the case of $(1,0)\oplus(0,1)$–representation space there is an analogy with the photon spin states. Namely, the operators X and Y correspond to the right and left polarization states of the photon. For that reason we will call the canonical basis consisting of the vectors $\mid lm;\dot{l}\dot{m}\rangle$ as *a helicity basis*. Infinitesimal operators of $SL(2,\mathbb{C})$ in the helicity basis have a very simple form

$$\begin{aligned}
\mathsf{A}_1 \mid l,m;\dot{l},\dot{m}\rangle &= -\frac{i}{2}\boldsymbol{\alpha}^l_m \mid l,m-1;\dot{l},\dot{m}\rangle - \frac{i}{2}\boldsymbol{\alpha}^l_{m+1} \mid l,m+1;\dot{l}\dot{m}\rangle,\\
\mathsf{A}_2 \mid l,m;\dot{l},\dot{m}\rangle &= \frac{1}{2}\boldsymbol{\alpha}^l_m \mid l,m-1;\dot{l},\dot{m}\rangle - \frac{1}{2}\boldsymbol{\alpha}^l_{m+1} \mid l,m+1;\dot{l},\dot{m}\rangle,\\
\mathsf{A}_3 \mid l,m;\dot{l},\dot{m}\rangle &= -im \mid l,m;\dot{l},\dot{m}\rangle,
\end{aligned} \tag{45}$$

$$\begin{aligned}
\mathsf{B}_1 \mid l,m;\dot{l},\dot{m}\rangle &= -\frac{1}{2}\boldsymbol{\alpha}^l_m \mid l,m-1;\dot{l},\dot{m}\rangle - \frac{1}{2}\boldsymbol{\alpha}^l_{m+1} \mid l,m+1;\dot{l},\dot{m}\rangle,\\
\mathsf{B}_2 \mid l,m;\dot{l},\dot{m}\rangle &= -\frac{i}{2}\boldsymbol{\alpha}^l_m \mid l,m-1;\dot{l},\dot{m}\rangle + \frac{i}{2}\boldsymbol{\alpha}^l_{m+1} \mid l,m+1;\dot{l},\dot{m}\rangle,\\
\mathsf{B}_3 \mid l,m;\dot{l},\dot{m}\rangle &= -m \mid l,m;\dot{l},\dot{m}\rangle,
\end{aligned} \tag{46}$$

$$\begin{aligned}
\widetilde{\mathsf{A}}_1 \mid l,m;\dot{l},\dot{m}\rangle &= -\frac{i}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}} \mid l,m;\dot{l},\dot{m}-1\rangle - \frac{i}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}+1} \mid l,m;\dot{l},\dot{m}+1\rangle,\\
\widetilde{\mathsf{A}}_2 \mid l,m;\dot{l},\dot{m}\rangle &= \frac{1}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}} \mid l,m;\dot{l},\dot{m}-1\rangle - \frac{1}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}+1} \mid l,m;\dot{l},\dot{m}+1\rangle,\\
\widetilde{\mathsf{A}}_3 \mid l,m;\dot{l},\dot{m}\rangle &= -i\dot{m} \mid l,m;\dot{l},\dot{m}\rangle,
\end{aligned} \tag{47}$$

$$\begin{aligned}
\widetilde{\mathsf{B}}_1 \mid l,m;\dot{l},\dot{m}\rangle &= \frac{1}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}} \mid l,m;\dot{l},\dot{m}-1\rangle + \frac{1}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}+1} \mid l,m;\dot{l},\dot{m}+1\rangle,\\
\widetilde{\mathsf{B}}_2 \mid l,m;\dot{l},\dot{m}\rangle &= \frac{i}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}} \mid l,m;\dot{l},\dot{m}-1\rangle - \frac{i}{2}\boldsymbol{\alpha}^{\dot{l}}_{\dot{m}+1} \mid l,m;\dot{l},\dot{m}+1\rangle,\\
\widetilde{\mathsf{B}}_3 \mid l,m;\dot{l},\dot{m}\rangle &= -\dot{m} \mid l,m;\dot{l},\dot{m}\rangle,
\end{aligned} \tag{48}$$

where

$$\boldsymbol{\alpha}^l_m = \sqrt{(l+m)(l-m+1)}.$$

The representation of the group $SL(2,\mathbb{C})$ in the space $\mathrm{Sym}(k,r)$ has a form

$$\begin{aligned}
T_g q(\mathfrak{z},\bar{\mathfrak{z}}) &= \frac{1}{z_1^k\,\bar{z}_1^r} T_g\left[z_1^k\bar{z}_1^r q\left(\frac{z_0}{z_1},\frac{\bar{z}_0}{\bar{z}_1}\right)\right] =\\
&= (\gamma\mathfrak{z}+\delta)^k(\overset{*}{\gamma}\bar{\mathfrak{z}}+\overset{*}{\delta})^r q\left(\frac{\alpha\mathfrak{z}+\beta}{\gamma\mathfrak{z}+\delta},\frac{\overset{*}{\alpha}\bar{\mathfrak{z}}+\overset{*}{\beta}}{\overset{*}{\gamma}\bar{\mathfrak{z}}+\overset{*}{\delta}}\right),
\end{aligned} \tag{49}$$

where

$$\mathfrak{z} = \frac{z_0}{z_1}, \quad \bar{\mathfrak{z}} = \frac{\bar{z}_0}{\bar{z}_1}.$$

In turn, every space $\mathrm{Sym}_{(k,r)}$ can be represented by a space of polynomials

$$p(z_0,z_1,\bar{z}_0,\bar{z}_1)=\sum_{\substack{(\alpha_1,\ldots,\alpha_k)\\(\dot{\alpha}_1,\ldots,\dot{\alpha}_r)}}\frac{1}{k!\,r!}a^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_r}z_{\alpha_1}\cdots z_{\alpha_k}\bar{z}_{\dot{\alpha}_1}\cdots\bar{z}_{\dot{\alpha}_r}. \tag{50}$$

$$(\alpha_i,\dot{\alpha}_i=0,1)$$

where the numbers $a^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_r}$ are unaffected at the permutations of indices. The expressions (50) can be understood as *functions on the Lorentz group.*

The infinitesimal operators A_i and B_i can be written via the complex Euler angles (36) as follows (for more details see [114])

$$\mathsf{A}_1 = \cos\psi^c\frac{\partial}{\partial\theta}+\frac{\sin\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varphi}-\cot\theta^c\sin\psi^c\frac{\partial}{\partial\psi}, \tag{51}$$

$$\mathsf{A}_2 = -\sin\psi^c\frac{\partial}{\partial\theta}+\frac{\cos\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varphi}-\cot\theta^c\cos\psi^c\frac{\partial}{\partial\psi}, \tag{52}$$

$$\mathsf{A}_3 = \frac{\partial}{\partial\psi}, \tag{53}$$

$$\mathsf{B}_1 = \cos\psi^c\frac{\partial}{\partial\tau}+\frac{\sin\psi^c}{\sin\theta^c}\frac{\partial}{\partial\epsilon}-\cot\theta^c\sin\psi^c\frac{\partial}{\partial\varepsilon}, \tag{54}$$

$$\mathsf{B}_2 = -\sin\psi^c\frac{\partial}{\partial\tau}+\frac{\cos\psi^c}{\sin\theta^c}\frac{\partial}{\partial\epsilon}-\cot\theta^c\cos\psi^c\frac{\partial}{\partial\varepsilon}, \tag{55}$$

$$\mathsf{B}_3 = \frac{\partial}{\partial\varepsilon}. \tag{56}$$

It is easy to verify that operators A_i, B_i, defined by the formulae (51)–(56), are satisfy the commutation relations (40).

Further, taking into account the expressions (51)–(56), we can write the operators (41) in the form

$$\mathsf{X}_1 = \cos\psi^c\frac{\partial}{\partial\theta^c}+\frac{\sin\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varphi^c}-\cot\theta^c\sin\psi^c\frac{\partial}{\partial\psi^c}, \tag{57}$$

$$\mathsf{X}_2 = -\sin\psi^c\frac{\partial}{\partial\theta^c}+\frac{\cos\psi^c}{\sin\theta^c}\frac{\partial}{\partial\varphi^c}-\cot\theta^c\cos\psi^c\frac{\partial}{\partial\psi^c}, \tag{58}$$

$$\mathsf{X}_3 = \frac{\partial}{\partial\psi^c}, \tag{59}$$

$$\mathsf{Y}_1 = \cos\dot{\psi}^c\frac{\partial}{\partial\dot{\theta}^c}+\frac{\sin\dot{\psi}^c}{\sin\dot{\theta}^c}\frac{\partial}{\partial\dot{\varphi}^c}-\cot\dot{\theta}^c\sin\dot{\psi}^c\frac{\partial}{\partial\dot{\psi}^c}, \tag{60}$$

$$\mathsf{Y}_2 = -\sin\dot{\psi}^c\frac{\partial}{\partial\dot{\theta}^c}+\frac{\cos\dot{\psi}^c}{\sin\dot{\theta}^c}\frac{\partial}{\partial\dot{\varphi}^c}-\cot\dot{\theta}^c\cos\dot{\psi}^c\frac{\partial}{\partial\dot{\psi}^c}, \tag{61}$$

$$\mathsf{Y}_3 = \frac{\partial}{\partial\dot{\psi}^c}, \tag{62}$$

where

$$\frac{\partial}{\partial\theta^c}=\frac{1}{2}\left(\frac{\partial}{\partial\theta}+i\frac{\partial}{\partial\tau}\right),\quad \frac{\partial}{\partial\varphi^c}=\frac{1}{2}\left(\frac{\partial}{\partial\varphi}+i\frac{\partial}{\partial\epsilon}\right),\quad \frac{\partial}{\partial\psi^c}=\frac{1}{2}\left(\frac{\partial}{\partial\psi}+i\frac{\partial}{\partial\varepsilon}\right),$$

$$\frac{\partial}{\partial\dot{\theta}^c}=\frac{1}{2}\left(\frac{\partial}{\partial\theta}-i\frac{\partial}{\partial\tau}\right),\quad \frac{\partial}{\partial\dot{\varphi}^c}=\frac{1}{2}\left(\frac{\partial}{\partial\varphi}-i\frac{\partial}{\partial\epsilon}\right),\quad \frac{\partial}{\partial\dot{\psi}^c}=\frac{1}{2}\left(\frac{\partial}{\partial\psi}-i\frac{\partial}{\partial\varepsilon}\right),$$

On the group $SL(2,\mathbb{C})$ there exist the following Laplace-Beltrami operators:

$$\begin{aligned}
\mathsf{X}^2 &= \mathsf{X}_1^2+\mathsf{X}_2^2+\mathsf{X}_3^2=\frac{1}{4}(\mathsf{A}^2-\mathsf{B}^2+2i\mathsf{A}\mathsf{B}),\\
\mathsf{Y}^2 &= \mathsf{Y}_1^2+\mathsf{Y}_2^2+\mathsf{Y}_3^2=\frac{1}{4}(\widetilde{\mathsf{A}}^2-\widetilde{\mathsf{B}}^2-2i\widetilde{\mathsf{A}}\widetilde{\mathsf{B}}).
\end{aligned}\tag{63}$$

At this point, we see that operators (63) contain the well known Casimir operators $\mathsf{A}^2-\mathsf{B}^2$, $\mathsf{A}\mathsf{B}$ of the Lorentz group. Substituting (57)-(62) into (63), we obtain an Euler parametrization of the Laplace-Beltrami operators:

$$\begin{aligned}
\mathsf{X}^2 &= \frac{\partial^2}{\partial\theta^{c2}}+\cot\theta^c\frac{\partial}{\partial\theta^c}+\frac{1}{\sin^2\theta^c}\left[\frac{\partial^2}{\partial\varphi^{c2}}-2\cos\theta^c\frac{\partial}{\partial\varphi^c}\frac{\partial}{\partial\psi^c}+\frac{\partial^2}{\partial\psi^{c2}}\right],\\
\mathsf{Y}^2 &= \frac{\partial^2}{\partial\dot{\theta}^{c2}}+\cot\dot{\theta}^c\frac{\partial}{\partial\dot{\theta}^c}+\frac{1}{\sin^2\dot{\theta}^c}\left[\frac{\partial^2}{\partial\dot{\varphi}^{c2}}-2\cos\dot{\theta}^c\frac{\partial}{\partial\dot{\varphi}^c}\frac{\partial}{\partial\dot{\psi}^c}+\frac{\partial^2}{\partial\dot{\psi}^{c2}}\right].
\end{aligned}\tag{64}$$

Matrix elements $t^l_{mn}(\mathfrak{g})=\mathfrak{M}^l_{mn}(\varphi^c,\theta^c,\psi^c)$ of irreducible representations of $SL(2,\mathbb{C})$ are eigenfunctions of the operators (64):

$$\begin{aligned}
\left[\mathsf{X}^2+l(l+1)\right]\mathfrak{M}^l_{mn}(\varphi^c,\theta^c,\psi^c) &= 0,\\
\left[\mathsf{Y}^2+\dot{l}(\dot{l}+1)\right]\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c,\dot{\theta}^c,\dot{\psi}^c) &= 0,
\end{aligned}\tag{65}$$

where

$$\begin{aligned}
\mathfrak{M}^l_{mn}(\varphi^c,\theta^c,\psi^c) &= e^{-i(m\varphi^c+n\psi^c)}Z^l_{mn}(\cos\theta^c),\\
\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c,\dot{\theta}^c,\dot{\psi}^c) &= e^{-i(\dot{m}\dot{\varphi}^c+\dot{n}\dot{\varphi}^c)}Z^{\dot{l}}_{\dot{m}\dot{n}}(\cos\dot{\theta}^c).
\end{aligned}\tag{66}$$

Substituting the hyperspherical functions (66) into (65) and taking into account the operators (64), we obtain

$$\begin{aligned}
\left[\frac{d^2}{d\theta^{c2}}+\cot\theta^c\frac{d}{d\theta^c}-\frac{m^2+n^2-2mn\cos\theta^c}{\sin^2\theta^c}+l(l+1)\right]Z^l_{mn}(\cos\theta^c) &= 0,\\
\left[\frac{d^2}{d\dot{\theta}^{c2}}+\cot\dot{\theta}^c\frac{d}{d\dot{\theta}^c}-\frac{\dot{m}^2+\dot{n}^2-2\dot{m}\dot{n}\cos\dot{\theta}^c}{\sin^2\dot{\theta}^c}+\dot{l}(\dot{l}+1)\right]Z^{\dot{l}}_{\dot{m}\dot{n}}(\cos\dot{\theta}^c) &= 0.
\end{aligned}$$

Finally, after substitutions $z=\cos\theta^c$ and $\overset{*}{z}=\cos\dot{\theta}^c$, we come to the following differential equations (a complex analog of the Legendre equations)

$$\left[(1-z^2)\frac{d^2}{dz^2}-2z\frac{d}{dz}-\frac{m^2+n^2-2mnz}{1-z^2}+l(l+1)\right]Z^l_{mn}(z) = 0,\tag{67}$$

$$\left[(1-\overset{*}{z}{}^2)\frac{d^2}{d\overset{*}{z}{}^2}-2\overset{*}{z}\frac{d}{d\overset{*}{z}}-\frac{\dot{m}^2+\dot{n}^2-2\dot{m}\dot{n}\overset{*}{z}}{1-\overset{*}{z}{}^2}+\dot{l}(\dot{l}+1)\right]Z^{\dot{l}}_{\dot{m}\dot{n}}(\overset{*}{z}) = 0.\tag{68}$$

The latter equations have three singular points -1, $+1$, ∞. Solutions of (67) have the form

$$Z^l_{mn}=\sum^l_{k=-l}i^{m-k}\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-k+1)\Gamma(l+k+1)}\times$$
$$\cos^{2l}\frac{\theta}{2}\tan^{m-k}\frac{\theta}{2}\times$$
$$\sum^{\min(l-m,l+k)}_{j=\max(0,k-m)}\frac{i^{2j}\tan^{2j}\dfrac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)}\times$$
$$\sqrt{\Gamma(l-n+1)\Gamma(l+n+1)\Gamma(l-k+1)\Gamma(l+k+1)}\cosh^{2l}\frac{\tau}{2}\tanh^{n-k}\frac{\tau}{2}\times$$
$$\sum^{\min(l-n,l+k)}_{s=\max(0,k-n)}\frac{\tanh^{2s}\dfrac{\tau}{2}}{\Gamma(s+1)\Gamma(l-n-s+1)\Gamma(l+k-s+1)\Gamma(n-k+s+1)}. \quad (69)$$

We will call the functions Z^l_{mn} in (69) as *hyperspherical functions*[1]. This form of the functions Z^l_{mn} immediately follows from the Cartan decompositions (38) and (39). Thus, matrix elements $t^l_{mn}(\mathfrak{g})$ are expressed by means of the function (*a generalized hyperspherical function*)

$$\mathfrak{M}^l_{mn}(\mathfrak{g})=e^{-m(\epsilon+i\varphi)}Z^l_{mn}(\cos\theta^c)e^{-n(\varepsilon+i\psi)}, \quad (70)$$

where

$$Z^l_{mn}(\cos\theta^c)=\sum^l_{k=-l}P^l_{mk}(\cos\theta)\mathfrak{P}^l_{kn}(\cosh\tau), \quad (71)$$

here $P^l_{mn}(\cos\theta)$ is a generalized spherical function on the group $SU(2)$ (see (8), (12)), and $\mathfrak{P}^l_{mn}(\cosh\tau)$ is an analog of the generalized spherical function for the group $SU(1,1)$ (see (25), (27)).

Further, using (10) and (28), we can write the hyperspherical functions (71) via the hypergeometric series:

$$Z^l_{mn}(\cos\theta^c)=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum^l_{k=-l}i^{m-k}\tan^{m-k}\frac{\theta}{2}\tanh^{n-k}\frac{\tau}{2}\times$$
$${}_2F_1\left(\begin{matrix}m-l+1,1-l-k\\ m-k+1\end{matrix}\;\middle|\; i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}n-l+1,1-l-k\\ n-k+1\end{matrix}\;\middle|\;\tanh^2\frac{\tau}{2}\right). \quad (72)$$

It is obvious that solutions $Z^{\dot{l}}_{\dot{m}\dot{n}}$ of the equation (68) have the same structure.

Further, from (72) we see that the function Z^l_{mn} depends on two variables θ and τ. Therefore, using Bateman factorization we can express the hyperspherical functions Z^l_{mn}

[1]The hyperspherical functions (or hyperspherical harmonics) are known in mathematics for a long time (see, for example, [10]). These functions are generalizations of the three-dimensional spherical functions on the case of n-dimensional euclidean spaces. For that reason we retain this name (hyperspherical functions) for the case of pseudo-euclidean spaces.

via Appell functions F_1–F_4 (hypergeometric series of two variables [2, 9]). For more details about this relationship see [117].

Using the formula (69), let us find explicit expressions for the matrices $T_l(\theta,\tau)$ of the finite–dimensional representations of $SL(2,\mathbb{C})$ at $l=0,\frac{1}{2},1$:

$$T_0(\theta,\tau)=1,$$

$$T_{\frac{1}{2}}(\theta,\tau)=\begin{pmatrix} Z^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}} & Z^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}} \\ Z^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}} & Z^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} \end{pmatrix}=$$

$$\begin{pmatrix} \cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2} & \cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2} \\ \cos\frac{\theta}{2}\sinh\frac{\tau}{2}+i\sin\frac{\theta}{2}\cosh\frac{\tau}{2} & \cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2} \end{pmatrix}, \tag{73}$$

$$T_1(\theta,\tau)=\begin{pmatrix} Z^1_{-1-1} & Z^1_{-10} & Z^1_{-11} \\ Z^1_{0-1} & Z^1_{00} & Z^1_{01} \\ Z^1_{1-1} & Z^1_{10} & Z^1_{11} \end{pmatrix}=$$

$$\left(\begin{matrix} \cos^2\frac{\theta}{2}\cosh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\sinh^2\frac{\tau}{2} & \frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau) \\ \frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau) & \cos\theta\cosh\tau+i\sin\theta\sinh\tau \\ \cos^2\frac{\theta}{2}\sinh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\cosh^2\frac{\tau}{2} & \frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau) \end{matrix}\right.$$

$$\left.\begin{matrix} \cos^2\frac{\theta}{2}\sinh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\cosh^2\frac{\tau}{2} \\ \frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+i\sin\theta\cosh\tau) \\ \cos^2\frac{\theta}{2}\cosh^2\frac{\tau}{2}+\frac{i\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\sinh^2\frac{\tau}{2} \end{matrix}\right). \tag{74}$$

4.1 Associated hyperspherical functions and a two-dimensional complex sphere

Let us consider now associated hyperspherical functions of the representation $T_\chi(\mathfrak{g})$, $\chi=(l,0)$, that is, the matrix elements $t^l_{m0}(\mathfrak{g})$ standing in one column with the function $t^l_{00}(\mathfrak{g})$. In this case from (70) we have

$$t^l_{m0}(\mathfrak{g})=\mathfrak{M}^l_{m0}(\mathfrak{g})=e^{-m(\epsilon+i\varphi)}Z^l_{m0}(\theta,\tau). \tag{75}$$

Hence it follows that matrix elements $t^l_{m0}(\mathfrak{g})$ do not depend on the Euler angles ε and ψ, that is, $t^l_{m0}(\mathfrak{g})$ are constant on the left adjacency classes formed by the subgroup Ω^c_ψ of the diagonal matrices $\begin{pmatrix} e^{\frac{i\psi^c}{2}} & 0 \\ 0 & e^{-\frac{i\psi^c}{2}} \end{pmatrix}$. Therefore,

$$t^l_{m0}(\mathfrak{g}h)=t^l_{m0}(\mathfrak{g}),\quad h\in\Omega^c_\psi.$$

We will denote the functions $Z^{m0}(\theta,\tau)$ via $Z_l^m(\theta,\tau)$. From (69) we obtain an explicit expression for the *associated hyperspherical function* $Z_l^m(\theta,\tau)$:

$$Z_l^m(\theta,\tau)=\sum_{k=-l}^{l} i^{m-k}\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-k+1)\Gamma(l+k+1)}\times$$
$$\cos^{2l}\frac{\theta}{2}\tan^{m-k}\frac{\theta}{2}\times$$
$$\sum_{j=\max(0,k-m)}^{\min(l-m,l+k)}\frac{i^{2j}\tan^{2j}\dfrac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)}\times$$
$$\Gamma(l+1)\sqrt{\Gamma(l-k+1)\Gamma(l+k+1)}\cosh^{2l}\frac{\tau}{2}\tanh^{-k}\frac{\tau}{2}\times$$
$$\sum_{s=\max(0,k)}^{\min(l,l+k)}\frac{\tanh^{2s}\dfrac{\tau}{2}}{\Gamma(s+1)\Gamma(l-s+1)\Gamma(l+k-s+1)\Gamma(s-k+1)}. \quad (76)$$

Further, from (72) it follows that

$$Z^l_{mn}=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l} i^{m-k}\tan^{m-k}\frac{\theta}{2}\tanh^{-k}\frac{\tau}{2}\times$$
$${}_2F_1\left(\begin{matrix}m-l+1,1-l-k\\ m-k+1\end{matrix}\;\middle|\; i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}-l+1,1-l-k\\ -k+1\end{matrix}\;\middle|\;\tanh^2\frac{\tau}{2}\right). \quad (77)$$

Associated hyperspherical functions admit a very elegant geometric interpretation, namely, they are the functions on the surface of the two-dimensional complex sphere. Indeed, let us construct in $\mathbb{C}^3$ the two–dimensional complex sphere from the quantities $z_k=x_k+iy_k$, $\overset{*}{z}_k=x_k-iy_k$ as follows (see Figure 1)

$$\mathbf{z}^2=z_1^2+z_2^2+z_3^2=\mathbf{x}^2-\mathbf{y}^2+2i\mathbf{x}\mathbf{y}=r^2 \quad (78)$$

and its complex conjugate (dual) sphere

$$\overset{*}{\mathbf{z}}{}^2=\overset{*}{z}_1{}^2+\overset{*}{z}_2{}^2+\overset{*}{z}_3{}^2=\mathbf{x}^2-\mathbf{y}^2-2i\mathbf{x}\mathbf{y}=\overset{*}{r}{}^2. \quad (79)$$

For more details about the two-dimensional complex sphere see [49, 50, 106]. It is well-known that both quantities $\mathbf{x}^2-\mathbf{y}^2$, $\mathbf{x}\mathbf{y}$ are invariant with respect to the Lorentz transformations, since a surface of the complex sphere is invariant (Casimir operators of the Lorentz group are constructed from such quantities, see also (63)). Moreover, since the real and imaginary parts of the complex two-sphere transform like the electric and magnetic fields, respectively, the invariance of $\mathbf{z}^2\sim(\mathbf{E}+i\mathbf{B})^2$ under proper Lorentz transformations is evident. At this point, the quantities $\mathbf{x}^2-\mathbf{y}^2$, $\mathbf{x}\mathbf{y}$ are similar to the well known electromagnetic invariants E^2-B^2, $\mathbf{EB}$. This intriguing relationship between the Laplace-Beltrami operators (63), Casimir operators of the Lorentz group and electromagnetic invariants $E^2-B^2\sim\mathbf{x}^2-\mathbf{y}^2$, $\mathbf{EB}\sim\mathbf{x}\mathbf{y}$ leads naturally to a Riemann-Silberstein representation of the electromagnetic field

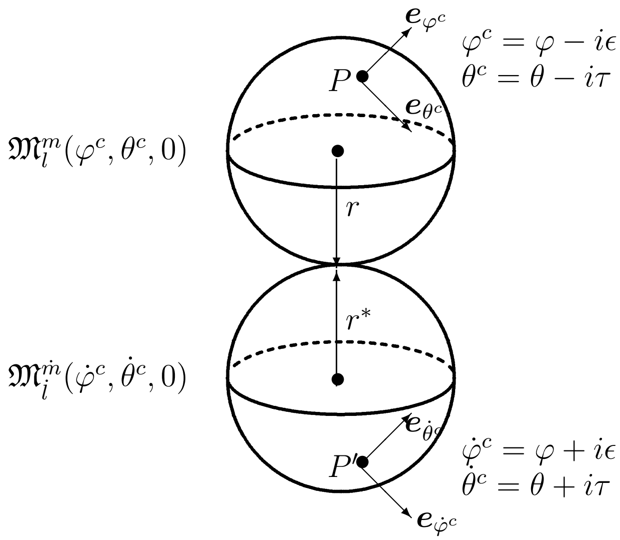

Figure 1 Two–dimensional complex sphere $z_1^2 + z_2^2 + z_3^2 = r^2$ in three–dimensional complex space $\mathbb{C}^3$. The space $\mathbb{C}^3$ is isometric to the bivector space $\mathbb{R}^6$. The dual (complex conjugate) sphere $\overset{*}{z}_1{}^2 + \overset{*}{z}_2{}^2 + \overset{*}{z}_3{}^2 = \overset{*}{r}{}^2$ is a mirror image of the complex sphere with respect to the hyperplane. The associated hyperspherical functions $\mathfrak{M}_l^m(\varphi^c, \theta^c, 0)$ ($\mathfrak{M}_{\dot{l}}^{\dot{m}}(\dot{\varphi}^c, \dot{\theta}^c, 0)$) are defined on the surface of the complex (dual) sphere.

(see the section 9). In other words, the two-dimensional sphere, considered as a homogeneous space of the Poincaré group, is the most suitable arena for the subsequent investigations in quantum electrodynamics. It is easy to see that three–dimensional complex space $\mathbb{C}^3$ is isometric to a real space $\mathbb{R}^{3,3}$ with a basis $\{i\mathbf{e}_1, i\mathbf{e}_2, i\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}$.

4.2 Matrix elements of principal and supplementary series of representations

As it has been shown in [78], for the case of principal unitary series representations of $SL(2,\mathbb{C})$ there exists an analog of the spinor representation formula (49):

$$T^\alpha f(z) = (a_{12}z + a_{22})^{\frac{\lambda}{2}+i\frac{\rho}{2}-1}\overline{(a_{12}z + a_{22})}^{-\frac{\lambda}{2}+i\frac{\rho}{2}-1} f\left(\frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}\right), \tag{80}$$

where $f(z)$ is a measurable functions of the Hilbert space $L_2(Z)$, satisfying the condition $\int |f(z)|^2 dz < \infty$, $z = x + iy$. At this point, the numbers l_0, l_1 and λ, ρ are related by the formulae

$$l_0 = \left|\frac{\lambda}{2}\right|, \quad l_1 = -i(\operatorname{sign}\lambda)\frac{\rho}{2} \quad \text{if } m \neq 0,$$
$$l_0 = 0, \quad l_1 = \pm i\frac{\rho}{2} \quad \text{if } m = 0.$$

A totality of all representations $a \to T^\alpha$, corresponding to all possible pairs λ, ρ, is called a principal series of representations of the group $SL(2,\mathbb{C})$. At this point, a comparison of (80) with the formula (49) for the spinor representation $\mathfrak{S}_l$ shows that the both formulas have the

same structure; only the exponents at the factors $(a_{12}z+a_{22})$, $\overline{(a_{12}z+a_{22})}$ and the functions $f(z)$ are different. In the case of spinor representations the functions $f(z)$ are polynomials $p(z,\bar{z})$ in the spaces $\mathrm{Sym}_{(k,r)}$, and in the case of a representation $\mathfrak{S}_{\lambda,\rho}$ of the principal series $f(z)$ are functions from the Hilbert space $L_2(Z)$.

Therefore, matrix elements of the principal series representations of the Lorentz group, making infinite-dimensional matrix, have the form

$$
\begin{gathered}
t_{mn}^{-\frac{1}{2}+i\rho}(\mathfrak{g})=e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}Z_{mn}^{-\frac{1}{2}+i\rho}=e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)}\times\\
\sum_{\lambda=-\infty}^{+\infty}\sum_{k=-\frac{\lambda}{2}}^{\left[\frac{\lambda}{2}\right]}i^{m-k}\sqrt{\Gamma(\tfrac{\lambda}{2}-m+1)\Gamma(\tfrac{\lambda}{2}+m+1)\Gamma(\tfrac{\lambda}{2}-k+1)\Gamma(\tfrac{\lambda}{2}+k+1)}\times\\
\cos^{\lambda}\frac{\theta}{2}\tan^{m-k}\frac{\theta}{2}\times\\
\sum_{j=\max(0,k-m)}^{\left[\min\left(\frac{\lambda}{2}-m,\frac{\lambda}{2}+k\right)\right]}\frac{i^{2j}\tan^{2j}\dfrac{\theta}{2}}{\Gamma(j+1)\Gamma(\frac{\lambda}{2}-m-j+1)\Gamma(\frac{\lambda}{2}+k-j+1)\Gamma(m-k+j+1)}\times\\
\sqrt{\Gamma(\tfrac{1}{2}+i\rho-n)\Gamma(\tfrac{1}{2}+i\rho+n)\Gamma(\tfrac{1}{2}+i\rho-k)\Gamma(\tfrac{1}{2}+i\rho+k)}\cosh^{-1+2i\rho}\frac{\tau}{2}\tanh^{n-k}\frac{\tau}{2}\times\\
\sum_{s=\max(0,k-n)}^{\left[\min\left(\frac{\lambda}{2}-n,\frac{\lambda}{2}+k\right)\right]}\frac{\tanh^{2s}\dfrac{\tau}{2}}{\Gamma(s+1)\Gamma(\frac{1}{2}+i\rho-n-s)\Gamma(\frac{1}{2}+i\rho+k-s)\Gamma(n-k+s+1)}. \quad (81)
\end{gathered}
$$

Thus, the matrix elements of the principal unitary series representations of the group $SL(2,\mathbb{C})$ are expressed via the function

$$
\mathfrak{M}_{mn}^{-\frac{1}{2}+i\rho}(\mathfrak{g})=e^{-m(\epsilon+i\varphi)}Z_{mn}^{-\frac{1}{2}+i\rho}(\cos\theta^c)e^{-n(\varepsilon+i\psi)}, \quad (82)
$$

where

$$
Z_{mn}^{-\frac{1}{2}+i\rho}(\cos\theta^c)=\sum_{\lambda=-\infty}^{+\infty}\sum_{k=-\frac{\lambda}{2}}^{\left[\frac{\lambda}{2}\right]}P_{mk}^{\frac{\lambda}{2}}(\cos\theta)\mathfrak{P}_{kn}^{-\frac{1}{2}+i\rho}(\cosh\tau).
$$

In the case of associated functions ($n=0$) we obtain

$$
Z_{-\frac{1}{2}+i\rho}^{m}(\cos\theta^c)=\sum_{\lambda=-\infty}^{+\infty}\sum_{k=-\frac{\lambda}{2}}^{\left[\frac{\lambda}{2}\right]}P_{mk}^{\frac{\lambda}{2}}(\cos\theta)\mathfrak{P}_{-\frac{1}{2}+i\rho}^{k}(\cosh\tau), \quad (83)
$$

where $\mathfrak{P}_{-\frac{1}{2}+i\rho}^{k}(\cosh\tau)$ are *conical functions* (see [9]). In this case our result agrees with the paper [33], where matrix elements (eigenfunctions of Casimir operators) of noncompact rotation groups are expressed in terms of conical and spherical functions (see also [120]).

Further, at $\lambda=0$ and $\rho=i\sigma$ from (80) it follows that

$$
T^{\alpha}f(z)=|a_{12}z+a_{22}|^{-2-\sigma}f\left(\frac{a_{11}z+a_{21}}{a_{12}z+a_{22}}\right).
$$

This formula defines an unitary representation $a \to T^{\alpha}$ of supplementary series $\mathfrak{D}_\sigma$ of the group $SL(2,\mathbb{C})$. At this point, for the supplementary series the relations

$$l_0 = 0, \quad l_1 = \pm\frac{\sigma}{2}$$

hold. In turn, the representation S_l of the group $SU(2)$ is contained in the representation $\mathfrak{D}_\sigma$ of supplementary series when l is an integer number. In this case S_l is contained in $\mathfrak{D}_\sigma$ exactly one time and the number $\frac{\lambda}{2} = 0$ is one from the set $-l, -l+1, \ldots, l$ [78].

Thus, matrix elements of supplementary series appear as a particular case of the matrix elements of the principal series at $\lambda = 0$ and $\rho = i\sigma$:

$$t^{-\frac{1}{2}-\sigma}_{mn}(\mathfrak{g}) = e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)} Z^{-\frac{1}{2}-\sigma}_{mn} = e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)} \times$$
$$\sqrt{\Gamma(\tfrac{1}{2}-\sigma-n)\Gamma(\tfrac{1}{2}-\sigma+n)\Gamma(\tfrac{1}{2}-\sigma-k)\Gamma(\tfrac{1}{2}-\sigma+k)}\cosh^{-1-2\sigma}\frac{\tau}{2}\tanh^{n-k}\frac{\tau}{2}\times$$
$$\sum^{\infty}_{s=\max(0,k-n)} \frac{\tanh^{2s}\dfrac{\tau}{2}}{\Gamma(s+1)\Gamma(\frac{1}{2}-\sigma-n-s)\Gamma(\frac{1}{2}-\sigma+k-s)\Gamma(n-k+s+1)}. \tag{84}$$

Or

$$\mathfrak{M}^{-\frac{1}{2}-\sigma}_{mn}(\mathfrak{g}) = e^{-m(\epsilon+i\varphi)}\mathfrak{P}^{-\frac{1}{2}-\sigma}_{mn}(\cosh\tau)e^{-n(\varepsilon+i\psi)},$$

that is, the hyperspherical function $Z^{-\frac{1}{2}+i\rho}_{mn}(\cos\theta^c)$ in the case of supplementary series is degenerated to the Jacobi function $\mathfrak{P}^{-\frac{1}{2}-\sigma}_{mn}(\cosh\tau)$. For the associated functions of supplementary series we obtain

$$\mathfrak{P}^{m}_{-\frac{1}{2}-\sigma}(\mathfrak{g}) = e^{-m(\epsilon+i\varphi)}\mathfrak{P}^{m}_{-\frac{1}{2}-\sigma}(\cosh\tau).$$

5 Harmonic analysis on the group $SL(2,\mathbb{C})$

First of all, on the group $SL(2,\mathbb{C})$ there exists an invariant measure $d\mathfrak{g}$, that is, such a measure that for any finite continuous function $f(\mathfrak{g})$ on $SL(2,\mathbb{C})$ the following equality

$$\int f(\mathfrak{g})d\mathfrak{g} = \int f(\mathfrak{g}_0\mathfrak{g})d\mathfrak{g} = \int f(\mathfrak{g}\mathfrak{g}_0)d\mathfrak{g} = \int f(\mathfrak{g}^{-1})d\mathfrak{g}$$

holds. Applying the equations (57)–(62), we express the Haar measure (left or right) in terms of the parameters (36):

$$d\mathfrak{g} = \sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon. \tag{85}$$

Thus, an invariant integration on the group $SL(2,\mathbb{C})$ is defined by the formula

$$\int\limits_{SL(2,\mathbb{C})} f(g)d\mathfrak{g} = \frac{1}{32\pi^4}\int\limits^{+\infty}_{-\infty}\int\limits^{+\infty}_{-\infty}\int\limits^{+\infty}_{-\infty}\int\limits^{2\pi}_{-2\pi}\int\limits^{2\pi}_{0}\int\limits^{\pi}_{0} f(\theta,\varphi,\psi,\tau,\epsilon,\varepsilon)\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon.$$

When we consider finite-dimensional (spinor) representations of $SL(2,\mathbb{C})$, we come naturally to a local isomorphism $SU(2)\otimes SU(2) \simeq SL(2,\mathbb{C})$ considered by many authors [50, 93]. Since

a dimension of the spinor representation $T_{\mathfrak{g}}$ of $SU(2)\otimes SU(2)$ is equal to $(2l+1)(2\dot{l}+1)$, then the functions $\sqrt{(2l+1)(2\dot{l}+1)}t^l_{mn}(\mathfrak{g})$ form a full orthogonal normalized system on this group with respect to the invariant measure $d\mathfrak{g}$. At this point, the index l runs all possible integer or half-integer non-negative values, and the indices m and n run the values $-l,-l+1,\ldots,l-1,l$. In virtue of (69) the matrix elements t^l_{mn} are expressed via the generalized hyperspherical function $t^l_{mn}(\mathfrak{g})=\mathfrak{M}^l_{mn}(\varphi,\epsilon,\theta,\tau,\psi,\varepsilon)$. Therefore,

$$\int\limits_{SU(2)\otimes SU(2)}\mathfrak{M}^l_{mn}(\mathfrak{g})\overline{\mathfrak{M}^l_{mn}(\mathfrak{g})}d\mathfrak{g}=\frac{32\pi^4}{(2l+1)(2\dot{l}+1)}\delta(\mathfrak{g}'-\mathfrak{g}),\tag{86}$$

where $\delta(\mathfrak{g}'-\mathfrak{g})$ is a δ-function on the group $SU(2)\otimes SU(2)$. An explicit form of δ-function is

$$\delta(\mathfrak{g}'-\mathfrak{g})=\delta(\varphi'-\varphi)\delta(\epsilon'-\epsilon)\delta(\cos\theta'\cosh\tau'-\cos\theta\cosh\tau)\times$$
$$\times\,\delta(\sin\theta'\sinh\tau'-\sin\theta\sinh\tau)\delta(\psi'-\psi)\delta(\varepsilon'-\varepsilon).$$

Substituting into (86) the expression

$$\mathfrak{M}^l_{mn}(\mathfrak{g})=e^{-m(\epsilon+i\varphi)}Z^l_{mn}e^{-n(\varepsilon+i\psi)}$$

and taking into account (85), we obtain

$$\int\limits_{SU(2)\otimes SU(2)}Z^l_{mn}\overline{Z^s_{pq}}e^{-(m+p)\epsilon}e^{-i(m-p)\varphi}\times$$
$$\times\,e^{-(n+q)\varepsilon}e^{-i(n-q)\psi}\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon=\frac{32\pi^4\delta_{ls}\delta_{mp}\delta_{nq}\delta(\mathfrak{g}'-\mathfrak{g})}{(2l+1)(2\dot{l}+1)}.$$

Thus, any square integrable function $f(\varphi^c,\theta^c,\psi^c)$ on the group $SU(2)\otimes SU(2)$, such that

$$\int\limits_{SU(2)\otimes SU(2)}|f(\varphi^c,\theta^c,\psi^c)|^2\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon<+\infty,$$

is expanded into a convergent (on an average) Fourier series on $SU(2)\otimes SU(2)$,

$$f(\varphi^c,\theta^c,\psi^c)=\sum^{\infty}_{l=0}\sum^{l}_{m=-l}\sum^{l}_{n=-l}\alpha^l_{mn}e^{-m(\epsilon+i\varphi)}Z^l_{mn}(\cos\theta^c)e^{-n(\varepsilon+i\psi)},\tag{87}$$

where

$$\alpha^l_{mn}=\frac{(-1)^{m-n}(2l+1)(2\dot{l}+1)}{32\pi^4}\times$$
$$\int\limits_{SU(2)\otimes SU(2)}f(\varphi^c,\theta^c,\psi^c)e^{i(m\varphi^c+n\psi^c)}Z^l_{mn}(\cos\theta^c)\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon.$$

The Parseval equality for the case of $SU(2)\otimes SU(2)$ is defined as follows

$$\sum^\infty_{l=0}\sum^l_{m=-l}\sum^l_{n=-l}|\alpha^l_{mn}|^2=$$
$$\frac{(2l+1)(2\dot{l}+1)}{32\pi^4}\int\limits_{SU(2)\otimes SU(2)}|f(\varphi^c,\theta^c,\psi^c)|^2\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon.$$

About convergence of Fourier series of the type (87) see [11].

In like manner we can define Fourier series on the two-dimensional complex sphere via the associated hyperspherical functions. An expansion of the functions on the surface of the two-dimensional sphere has an important meaning for the subsequent physical applications.

So, let $f(\varphi^c,\theta^c)$ be a function on the complex two-sphere $\mathbb{S}^2$, such that

$$\int\limits_{\mathbb{S}^2}|f(\varphi^c,\theta^c)|^2\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\tau d\epsilon<+\infty,$$

then $f(\varphi^c,\theta^c)$ is expanded into a convergent Fourier series on $\mathbb{S}^2$,

$$f(\varphi^c,\theta^c)=\sum^\infty_{l=0}\sum^l_{m=-l}\alpha^l_m e^{-m(\epsilon+i\varphi)}Z^m_l(\cos\theta^c),$$

where

$$\alpha^l_m=\frac{(-1)^m(2l+1)(2\dot{l}+1)}{32\pi^4}\int\limits_{\mathbb{S}^2}f(\varphi^c,\theta^c)e^{im\varphi^c}Z^m_l(\cos\theta^c)\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\tau d\epsilon,$$

and $Z^m_l(\cos\theta^c)$ is an associated hyperspherical function, $d\mathfrak{g}=\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\tau d\epsilon$ is a Haar measure on the sphere $\mathbb{S}^2$. Correspondingly, the Parseval equality on $\mathbb{S}^2$ has a form

$$\sum^\infty_{l=0}\sum^l_{m=-l}|\alpha^l_m|^2=\frac{(2l+1)(2\dot{l}+1)}{32\pi^4}\int\limits_{\mathbb{S}^2}|f(\varphi^c,\theta^c)|^2\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\tau d\epsilon.$$

In the case of infinite-dimensional representations of $SL(2,\mathbb{C})$ we come to the Fourier integrals on the Lorentz group. Let $f(\mathfrak{g})$ be a square integrable function on $SL(2,\mathbb{C})$ and let

$$T^\alpha(f)=\int\limits_{SL(2,\mathbb{C})}f(\mathfrak{g})T^\alpha(\mathfrak{g})d\mathfrak{g} \tag{88}$$

be a Fourier transform for all representations of the principal unitary series, then the inverse Fourier transform is defined by a formula

$$f(\mathfrak{g})=\frac{1}{16\pi^4}\sum^{+\infty}_{\lambda=-\infty}\int\limits^{+\infty}_0\mathrm{Tr}\left[T^{\rho,\lambda}(f)(T^{\rho,\lambda}(\mathfrak{g}))^*\right](\lambda^2+\rho^2)d\rho, \tag{89}$$

where the integral in the right part is convergent on an average, and the representation $T^{\alpha}=T^{\rho,\lambda}$ is defined by the formula (80). There is an analog of the Plancherel formula,

$$\int\limits_{SL(2,\mathbb{C})}|f(\mathfrak{g})|^2 d\mathfrak{g}=\frac{1}{16\pi^4}\sum^{+\infty}_{\lambda=-\infty}\int\limits^{+\infty}_{0}\operatorname{Tr}\left[(T^{\rho,\lambda}(f))^{*}T^{\rho,\lambda}(f)\right](\lambda^2+\rho^2)d\rho, \tag{90}$$

where $d\mathfrak{g}$ is the Haar measure on $SL(2,\mathbb{C})$. The integrals and series in the right part of (90) are convergent absolutely. Fourier integrals on the Lorentz group were studied by many authors [78, 97, 98, 18, 86, 37, 119, 125, 64, 67, 65, 47, 48, 91, 123, 126, 127]. It is obvious that the formulae (88)–(90) can be expressed via the matrix elements (81).

6 Fields on the Poincaré group

Fields on the Poincaré group present itself a natural generalization of the concept of wave function. These fields (generalized wave functions) were introduced independently by several authors [40, 7, 128, 100] mainly in connection with constructing relativistic wave equations (a so called Z-description of the relativistic spin [43]). The following logical step was done by Finkelstein [32], he suggested to consider the wave function depending both the coordinates on the Minkowski spacetime and some continuous variables corresponding to spin degrees of freedom (internal space). In essence, this generalization consists in replacing the Minkowski space by a larger space on which the Poincaré group acts. If this action is to be transitive, one is lead to consider the homogeneous spaces of the Poincaré group. All the homogeneous spaces of this type were listed by Finkelstein [32] and by Bacry and Kihlberg [5] and the fields on these spaces were considered in the works [69, 6, 80, 57, 58, 111, 42].

A homogeneous space $\mathcal{M}$ of a group G has the following properties:
a) It is a topological space on which the group G acts continuously, that is, let y be a point in $\mathcal{M}$, then gy is defined and is again a point in $\mathcal{M}$ $(g\in G)$.
b) This action is transitive, that is, for any two points y_1 and y_2 in $\mathcal{M}$ it is always possible to find a group element $g\in G$ such that $y_2=gy_1$.
There is a one-to-one correspondence between the homogeneous spaces of G and the coset spaces of G. Let H_0 be a maximal subgroup of G which leaves the point y_0 invariant, $gy_0=y_0$, $g\in H_0$, then H_0 is called the stabilizer of y_0. Representing now any group element of G in the form $g=g_cg_0$, where $g_0\in H_0$ and $g_c\in G/H_0$, we see that, by virtie of the transitivity property, any point $y\in\mathcal{M}$ can be given by $y=g_cg_0y_0=g_cy$. Hence it follows that the elements g_c of the coset space give a parametrization of $\mathcal{M}$. The mapping $\mathcal{M}\leftrightarrow G/H_0$ is continuous since the group multiplication is continuous and the action on $\mathcal{M}$ is continuous by definition. The stabilizers H and H_0 of two different points y and y_0 are conjugate, since from $H_0g_0=g_0$, $y_0=g^{-1}y$, it follows that $gH_0g^{-1}y=y$, that is, $H=gH_0g^{-1}$.

Coming back to the Poincaré group $\mathcal{P}$, we see that the enumeration of the different homogeneous spaces $\mathcal{M}$ of $\mathcal{P}$ amounts to an enumeration of the subgroups of $\mathcal{P}$ up to a conjugation. Following to Finkelstein, we require that $\mathcal{M}$ always contains the Minkowski space $\mathbb{R}^{1,3}$ which means that four parameters of $\mathcal{M}$ can be denoted by $x\,(x^{\mu})$. This means that the stabilizer H of a given point in $\mathcal{M}$ can never contain an element of the translation subgroup of $\mathcal{P}$. Thus, the stabilizer must be a subgroup of the homogeneous Lorentz group $\mathfrak{G}_+$.

In such a way, studying different subgroups of $\mathfrak{G}_+$, we obtain a full list of homogeneous spaces $\mathcal{M} = \mathcal{P}/H$ of the Poincaré group. In the present paper we restrict ourselves by a consideration of the following four homogeneous spaces:

$$\begin{array}{rcl} \mathcal{M}_{10} & = & \mathbb{R}^{1,3} \times \mathfrak{L}_6, \quad H = 0; \\ \mathcal{M}_8 & = & \mathbb{R}^{1,3} \times \mathbb{S}^2, \quad H = \Omega^c_\psi; \\ \mathcal{M}_7 & = & \mathbb{R}^{1,3} \times \mathbb{H}^3, \quad H = SU(2); \\ \mathcal{M}_6 & = & \mathbb{R}^{1,3} \times S^2, \quad H = \{\Omega^c_\psi, \Omega_\tau, \Omega_\epsilon\}. \end{array}$$

Hence it follows that a group manifold of the Poincaré group, $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathfrak{L}_6$, is a maximal homogeneous space of $\mathcal{P}$, $\mathfrak{L}_6$ is a group manifold of the Lorentz group. The fields on the manifold $\mathcal{M}_{10}$ were considered by Lurçat [69]. These fields depend on all the ten parameters of $\mathcal{P}$:

$$\boldsymbol{\psi}(x, \mathfrak{g}) = \psi(x)\psi(\mathfrak{g}) = \psi(x_0, x_1, x_2, x_3)\psi(\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_5, \mathfrak{g}_6),$$

where an explicit form of $\psi(x)$ is given by the exponentials, and the functions $\psi(\mathfrak{g})$ are expressed via the generalized hyperspherical functions $\mathfrak{M}^l_{mn}(\mathfrak{g})$ (see (70)) in the case of finite dimensional representations and via the function (81) in the case of principal series of unitary representations.

The following eight-dimensional homogeneous space $\mathcal{M}_8 = \mathbb{R}^{1,3} \times \mathbb{S}^2$ is a direct product of the Minkowski space $\mathbb{R}^{1,3}$ and the complex two-sphere $\mathbb{S}^2$. In this case the stabilizer H consists of the subgroup Ω^c_ψ of the diagonal matrices $\begin{pmatrix} e^{\frac{i\psi^c}{2}} & 0 \\ 0 & e^{-\frac{i\psi^c}{2}} \end{pmatrix}$. Bacry and Kihlberg [5] claimed that the space $\mathcal{M}_8$ is the most suitable for a description of both half-integer and integer spins. The fields, defined in $\mathcal{M}_8$, depend on the eight parameters of $\mathcal{P}$:

$$\boldsymbol{\psi}(x, \varphi^c, \theta^c) = \psi(x)\psi(\varphi^c, \theta^c) = \psi(x_0, x_1, x_2, x_3)\psi(\varphi, \epsilon, \theta, \tau), \tag{91}$$

where the functions $\psi(\varphi^c, \theta^c)$ are expressed via the associated hyperspherical functions (75) defined on the surface of the complex two-sphere $\mathbb{S}^2$.

In turn, a seven-dimensional homogeneous space $\mathcal{M}_7 = \mathbb{R}^{1,3} \times \mathbb{H}^3$ is a direct product of $\mathbb{R}^{1,3}$ and a three-dimensional timelike (two sheeted) hyperboloid $\mathbb{H}^3$. The stabilizer H consists of the subgroup of three-dimensional rotations, $SU(2)$. Quantum field theory on the space $\mathcal{M}_7$ was studied by Boyer and Fleming [15]. They showed that the fields built over $\mathcal{M}_7$ are in general nonlocal and become local only when the finite dimensional representations of the Lorentz group are used. It is easy to see that the fields $\boldsymbol{\psi} \in \mathcal{M}_7$ depend on the seven parameters of the Poincaré group:

$$\boldsymbol{\psi}(x, \tau, \epsilon, \varepsilon) = \psi(x_0, x_1, x_2, x_3)\psi(\tau, \epsilon, \varepsilon),$$

where the functions $\psi(\tau, \epsilon, \varepsilon)$ are expressed via $e^{-(m\epsilon+n\varepsilon)}\mathfrak{P}^l_{mn}(\cosh\tau)$ in the case of finite dimensional representations (the function $\mathfrak{P}^l_{mn}(\cosh\tau)$ is of the type (27)), and also via $e^{-(m\epsilon+n\varepsilon)}\mathfrak{P}^{-\frac{1}{2}+i\rho}_{mn}(\cosh\tau)$ in the case of principal series of unitary representations, and via $e^{-(m\epsilon+n\varepsilon)}\mathfrak{P}^{l,\pm}_{mn}(\cosh\tau)$ in the case of the discrete series.

Further, a six-dimensional space $\mathcal{M}_6 = \mathbb{R}^{1,3} \times S^2$ is a minimal homogeneous space of the Poincaré group, since the real two-sphere S^2 has a minimal possible dimension among the homogeneous spaces of the Lorentz group. In this case, the stabilizer H consists of the

subgroup Ω^c_ψ and the subgroups Ω_τ and Ω_ϵ formed by the matrices $\begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{pmatrix}$ and $\begin{pmatrix} e^{\frac{\epsilon}{2}} & 0 \\ 0 & e^{-\frac{\epsilon}{2}} \end{pmatrix}$, respectively. It is not hard to see that the two-dimensional real sphere coincides with a well known Penrose's celestial sphere S^- (or anti-celestial sphere S^+) [82], and by this reason we can define the two types of $\mathcal{M}_6$, namely, $\mathcal{M}^\pm_6 = \mathbb{R}^{1,3} \times S^\pm$. Field models on the homogeneous space $\mathcal{M}_6$ have been considered in recent works [63, 70, 28]. In the paper [28] Drechsler considered the real two-sphere as a 'spin shell' $S^2_{r=2s}$ of radius $r = 2s$, where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The fields, defined in $\mathcal{M}_6$, depend on the six parameters of $\mathcal{P}$:

$$\boldsymbol{\psi}(x,\varphi,\theta) = \psi(x_0,x_1,x_2,x_3)\psi(\varphi,\theta),$$

where the functions $\psi(\varphi,\theta)$ are expressed via the generalized spherical functions of the type $e^{-im\varphi}P^l_{m0}(\cos\theta)$ or via the Wigner D-functions.

6.1 Harmonic analysis on $SU(2) \otimes SU(2) \odot T_4$

In this subsection we will study Fourier series on the Poincaré group $\mathcal{P}$. First of all, the group $\mathcal{P}$ has the same number of connected components as with the Lorentz group. Later on we will consider only the component $\mathcal{P}^\uparrow_+$ corresponding the connected component $L^\uparrow_+$ (so called special Lorentz group, see [92]). As is known, an universal covering $\overline{\mathcal{P}^\uparrow_+}$ of the group $\mathcal{P}^\uparrow_+$ is defined by a semidirect product $\overline{\mathcal{P}^\uparrow_+} = SL(2,\mathbb{C}) \odot T_4 \simeq \mathbf{Spin}_+(1,3) \odot T_4$, where T_4 is a subgroup of four-dimensional translations. Since the Poincaré group is a 10-parameter group, then an invariant measure on this group has a form

$$d^{10}\boldsymbol{\alpha} = d^6\mathfrak{g}d^4x,$$

where $d^6\mathfrak{g}$ is the Haar measure on the Lorentz group. Or, taking into account (85), we obtain

$$d\boldsymbol{\alpha} = \sin\theta^c \sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon dx_1 dx_2 dx_3 dx_4, \tag{92}$$

where $x_i \in T_4$.

Thus, an invariant integration on the group $SL(2,\mathbb{C}) \odot T_4$ is defined by a formula

$$\int\limits_{SL(2,\mathbb{C})\odot T_4} f(\boldsymbol{\alpha})d^{10}\boldsymbol{\alpha} = \int\limits_{SL(2,\mathbb{C})}\int\limits_{T_4} f(x,\mathfrak{g})d^4x d^6\mathfrak{g},$$

where $f(\boldsymbol{\alpha})$ is a finite continuous function on $SL(2,\mathbb{C}) \odot T_4$.

In the case of finite-dimensional representations we come again to a local isomorphism $SL(2,\mathbb{C}) \odot T_4 \simeq SU(2) \otimes SU(2) \odot T_4$. In this case basis representation functions of the Poincaré group are defined by symmetric polynomials of the form

$$p(x,z,\bar{z}) = \sum_{\substack{(\alpha_1,\ldots,\alpha_k) \\ (\dot{\alpha}_1,\ldots,\dot{\alpha}_r)}} \frac{1}{k!\,r!} a^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_r}(x) z_{\alpha_1}\cdots z_{\alpha_k}\bar{z}_{\dot{\alpha}_1}\cdots\bar{z}_{\dot{\alpha}_r}, \tag{93}$$

$$(\alpha_i, \dot{\alpha}_i = 0, 1)$$

where the coefficients $a^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_r}$ depend on the variables x^α $(\alpha = 0,1,2,3)$ (the parameters of T_4). The functions (93) should be considered as *the functions on the Poincaré group*. Some

applications of these functions contained in [42]. The group T_4 is an Abelian group formed by a direct product of the four one-dimensional translation groups, T_1, where T_1 is isomorphic to an additive group of real numbers $\mathbb{R}$ (usual Fourier analysis is formulated in terms of the group $\mathbb{R}$). Hence it follows that all irreducible representations of T_4 are one-dimensional and expressed via the exponentials. Thus, the basis functions (matrix elements) of the finite-dimensional representations of $\mathcal{P}$ have the form

$$t^l_{mn}(\boldsymbol{\alpha}) = e^{-ipx}\mathfrak{M}^l_{mn}(\mathfrak{g}), \tag{94}$$

where $x = (x_1, x_2, x_3, x_4)$, and $\mathfrak{M}^l_{mn}(\mathfrak{g})$ is the generalized hyperspherical function (70).

Let us consider now the configuration space $\mathcal{M}_8 = \mathbb{R}^{1,3} \times \mathbb{S}^2$. In this case the Fourier series on $\mathcal{M}_8$ can be defined as follows

$$f(\boldsymbol{\alpha}) = \sum_{p=-\infty}^{+\infty} e^{ipx} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \alpha^l_m \mathfrak{M}^m_l(\varphi, \epsilon, \theta, \tau, 0, 0), \tag{95}$$

where

$$\alpha^l_m = \frac{(-1)^m(2l+1)(2\dot{l}+1)}{32\pi^4} \int\limits_{\mathbb{S}^2}\int\limits_{T_4} f(\boldsymbol{\alpha})e^{-ipx}\mathfrak{M}^m_l(\varphi, \epsilon, \theta, \tau, 0, 0)d^4x d^4\mathfrak{g},$$

and $d^4\mathfrak{g} = \sin\theta^c \sin\dot{\theta}^c d\theta d\varphi d\tau d\epsilon$ is the Haar measure on $\mathbb{S}^2$, $f(\boldsymbol{\alpha})$ is the square integrable function on $\mathcal{M}_8$, such that

$$\int\limits_{\mathbb{S}^2}\int\limits_{T_4} |f(\boldsymbol{\alpha})|^2 d^4x d^4\mathfrak{g} < +\infty.$$

Coming to the space $\mathcal{M}_7 = \mathbb{R}^{1,3} \times \mathbb{H}^3$ and restricting by the finite dimensional representations, we see that the Fourier series on $\mathcal{M}_7$ can be defined as

$$f(x, \tau, \epsilon, \varepsilon) = \sum_{p=-\infty}^{+\infty} e^{ipx} \sum_{l=0}^{\infty}\sum_{m,n=-l}^{l} \alpha^l_{mn} e^{-(m\epsilon+n\varepsilon)}\mathfrak{P}^l_{mn}(\cosh\tau),$$

where

$$\alpha^l_{mn} = \frac{(-1)^{m-n}(2l+1)}{16\pi^2} \int\limits_{SU(1,1)}\int\limits_{T_4} f(x, \tau, \epsilon, \varepsilon)e^{-ipx}\mathfrak{P}^l_{mn}(\cosh\tau)e^{m\epsilon+n\varepsilon}d^4x d^3g,$$

and $d^3g = \sinh\tau d\epsilon d\tau d\varepsilon$ is the Haar measure on $SU(1,1)$, $f(x, \tau, \epsilon, \varepsilon)$ is the square integrable function on $\mathcal{M}_7$. In the case of infinite dimensional representations (principal and discrete series) we come to the Fourier integrals on $\mathcal{M}_7$, where the direct and inverse Fourier transforms are of the type (31)–(35). The consideration of the infinite dimensional representations leads immediately to a harmonic analysis on the hyperboloids [107]. However, all the fields built in terms of the Fourier integrals on $\mathcal{M}_7$ are in general nonlocal. The same statement holds also for the fields built in terms of the Fourier integrals on $\mathcal{M}_{10}$ and $\mathcal{M}_8$, where the direct and inverse Fourier transforms are defined by the formulae (88)–(90).

Further, the Fourier series on the homogeneous space $\mathcal{M}_6 = \mathbb{R}^{1,3} \times S^2$ are defined like the series on the group $SU(2)$ considered in the subsection 2.1. For the physical purposes

it is useful to express the fields on $\mathcal{M}_6$ via the Wigner D-functions. Replacing the basis (7) by

$$f_n(\mathfrak{z}) = i^n \psi_n(\mathfrak{z}), \quad n = -l, -l+1, \ldots, l, \tag{96}$$

we obtain matrix elements $D^l_{mn}(\theta)$ of the operators $T_l(\theta)$ related with the matrix elements $t^l_{mn}(\theta)$ by a formula

$$D^l_{mn}(\theta) = i^{n-m} t^l_{mn}(\theta) = i^{n-m} P^l_{mn}(\cos\theta).$$

In contrast to $P^l_{mn}(\cos\theta)$, the functions $D^l_{mn}(\theta)$ are real. The full matrix element in the basis (96) has a form

$$D^l_{mn}(u) = e^{-i(m\varphi+n\psi)} D^l_{mn}(\theta).$$

Therefore, for any square integrable function $f(x,\varphi,\theta)$ on $\mathcal{M}_6$ we have

$$f(x,\varphi,\theta) = \sum_{p=-\infty}^{+\infty} e^{ipx} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \alpha^l_m e^{-im\varphi} D^l_{m0}(\theta),$$

where

$$\alpha^l_m = \frac{(-1)^m(2l+1)}{4\pi} \int\limits_{S^2} \int\limits_{T_4} e^{-ipx} f(x,\varphi,\theta) e^{im\varphi} D^l_{m0}(\theta) d^4 d^2\boldsymbol{\xi},$$

and $d^2\boldsymbol{\xi} = 1/4\pi \sin\theta d\theta d\varphi$ is an invariant measure on the sphere S^2. In such a way we come here to a Fourier analysis on the sphere [99]. It should be noted that solutions of relativistic wave equations and quantization procedures was studied by Malin [72, 73] in terms of the functions over the group $SU(2)$.

7 Lagrangian formalism and field equations on the Poincaré group

We will start with a more general homogeneous space of the group $\mathcal{P}$, $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathfrak{L}_6$ (group manifold of the Poincaré group). Let $\mathcal{L}(\boldsymbol{\alpha})$ be a Lagrangian on the group manifold $\mathcal{M}_{10}$ (in other words, $\mathcal{L}(\boldsymbol{\alpha})$ is a 10-dimensional point function), where $\boldsymbol{\alpha}$ is the parameter set of this group. Then an integral for $\mathcal{L}(\boldsymbol{\alpha})$ on some 10-dimensional volume Ω of the group manifold we will call *an action on the Poincaré group*:

$$A = \int\limits_{\Omega} d\boldsymbol{\alpha} \mathcal{L}(\boldsymbol{\alpha}),$$

where $d\boldsymbol{\alpha}$ is a Haar measure on the group $\mathcal{P}$ (see (92)).

Let $\boldsymbol{\psi}(\boldsymbol{\alpha})$ be a function on the group manifold $\mathcal{M}_{10}$ (now it is sufficient to assume that $\boldsymbol{\psi}(\boldsymbol{\alpha})$ is a square integrable function on the Poincaré group) and let

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\psi}} - \frac{\partial}{\partial \boldsymbol{\alpha}} \frac{\partial \mathcal{L}}{\partial \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{\alpha}}} = 0 \tag{97}$$

be Euler-Lagrange equations on $\mathcal{M}_{10}$ (more precisely speaking, the equations (97) act on the tangent bundle $T\mathcal{M}_{10} = \bigcup\limits_{\boldsymbol{\alpha}\in\mathcal{M}_{10}} T_{\boldsymbol{\alpha}}\mathcal{M}_{10}$ of the manifold $\mathcal{M}_{10}$, see [3]). Let us introduce a Lagrangian $\mathcal{L}(\boldsymbol{\alpha})$ depending on the field function $\boldsymbol{\psi}(\boldsymbol{\alpha})$ as follows

$$\mathcal{L}(\boldsymbol{\alpha}) = -\frac{1}{2}\left(\boldsymbol{\psi}^*(\boldsymbol{\alpha}) B_\mu \frac{\partial \boldsymbol{\psi}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}_\mu} - \frac{\partial \boldsymbol{\psi}^*(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}_\mu} B_\mu \boldsymbol{\psi}(\boldsymbol{\alpha}) \right) - \kappa \boldsymbol{\psi}^*(\boldsymbol{\alpha}) B_{11} \boldsymbol{\psi}(\boldsymbol{\alpha}),$$

where B_ν ($\nu = 1, 2, \ldots, 10$) are square matrices. The number of rows and columns in these matrices is equal to the number of components of $\boldsymbol{\psi}(\boldsymbol{\alpha})$, κ is a non-null real constant.

Further, if B_{11} is non-singular, then we can introduce the matrices

$$\Gamma_\mu = B_{11}^{-1} B_\mu, \quad \mu = 1, 2, \ldots, 10,$$

and represent the Lagrangian $\mathcal{L}(\boldsymbol{\alpha})$ in the form

$$\mathcal{L}(\boldsymbol{\alpha}) = -\frac{1}{2}\left(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\Gamma_\mu \frac{\partial \boldsymbol{\psi}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}_\mu} - \frac{\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}_\mu}\Gamma_\mu \boldsymbol{\psi}(\boldsymbol{\alpha})\right) - \kappa \overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}), \tag{98}$$

where

$$\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}) = \boldsymbol{\psi}^*(\boldsymbol{\alpha}) B_{11}.$$

Varying independently $\psi(x)$ and $\overline{\psi}(x)$, we obtain from (98) in accordance with (97) the following equations:

$$\begin{array}{rcl} \Gamma_i \dfrac{\partial \psi(x)}{\partial x_i} + \kappa \psi(x) & = & 0, \\ \Gamma_i^T \dfrac{\partial \overline{\psi}(x)}{\partial x_i} - \kappa \overline{\psi}(x) & = & 0. \end{array} \qquad (i = 1, \ldots, 4) \tag{99}$$

Analogously, varying independently $\psi(\mathfrak{g})$ and $\overline{\psi}(\mathfrak{g})$ one gets

$$\begin{array}{rcl} \Gamma_k \dfrac{\partial \psi(\mathfrak{g})}{\partial \mathfrak{g}_k} + \kappa \psi(\mathfrak{g}) & = & 0, \\ \Gamma_k^T \dfrac{\overline{\psi}(\mathfrak{g})}{\partial \mathfrak{g}_k} - \kappa \overline{\psi}(\mathfrak{g}) & = & 0, \end{array} \qquad (k = 1, \ldots, 6) \tag{100}$$

where

$$\psi(\mathfrak{g}) = \begin{pmatrix} \psi(\mathfrak{g}) \\ \dot{\psi}(\mathfrak{g}) \end{pmatrix}, \quad \Gamma_k = \begin{pmatrix} 0 & \Lambda_k^* \\ \Lambda_k & 0 \end{pmatrix}.$$

The doubling of representations, described by a bispinor $\psi(\mathfrak{g}) = (\psi(\mathfrak{g}), \dot{\psi}(\mathfrak{g}))^T$, is the well known feature of the Lorentz group representations [36, 78]. Since an universal covering $SL(2, \mathbb{C})$ of the proper orthochronous Lorentz group is a complexification of the group $SU(2)$ (see the section 4), then it is more convenient to express six parameters $\mathfrak{g}_k$ of the Lorentz group via three parameters a_1, a_2, a_3 of the group $SU(2)$. It is obvious that $\mathfrak{g}_1 = a_1$, $\mathfrak{g}_2 = a_2$, $\mathfrak{g}_3 = a_3$, $\mathfrak{g}_4 = ia_1$, $\mathfrak{g}_5 = ia_2$, $\mathfrak{g}_6 = ia_3$. Then the first equation from (100) can be written as

$$\begin{array}{rcl} \displaystyle\sum_{j=1}^{3} \Lambda_j^* \frac{\partial \dot{\psi}}{\partial \widetilde{a}_j} + i \sum_{j=1}^{3} \Lambda_j^* \frac{\partial \dot{\psi}}{\partial \widetilde{a}_j^*} + \kappa^c \psi & = & 0, \\ \displaystyle\sum_{j=1}^{3} \Lambda_j \frac{\partial \psi}{\partial a_j} - i \sum_{j=1}^{3} \Lambda_j \frac{\partial \psi}{\partial a_j^*} + \dot{\kappa}^c \dot{\psi} & = & 0, \end{array} \tag{101}$$

where $a_1^* = -i\mathfrak{g}_4$, $a_2^* = -i\mathfrak{g}_5$, $a_3^* = -i\mathfrak{g}_6$, and $\widetilde{a}_j$, $\widetilde{a}_j^*$ are the parameters corresponding the dual basis. In essence, the equations (101) are defined in a three-dimensional complex space $\mathbb{C}^3$. In turn, the space $\mathbb{C}^3$ is isometric to a 6-dimensional bivector space $\mathbb{R}^6$ (a parameter space of the Lorentz group [56, 84]). The bivector space $\mathbb{R}^6$ is a tangent space of the group

manifold $\mathfrak{L}_6$ of the Lorentz group, that is, the manifold $\mathfrak{L}_6$ in the each its point is equivalent locally to the space $\mathbb{R}^6$. Thus, for all $\mathfrak{g} \in \mathfrak{L}_6$ we have $T_{\mathfrak{g}}\mathfrak{L}_6 \simeq \mathbb{R}^6$. There exists a close relationship between the metrics of the Minkowski spacetime $\mathbb{R}^{1,3}$ and the metrics of $\mathbb{R}^6$ defined by the formulae (see [84])

$$g_{ab} \longrightarrow g_{\alpha\beta\gamma\delta} \equiv g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}, \tag{102}$$

where $g_{\alpha\beta}$ is a metric tensor of the spacetime $\mathbb{R}^{1,3}$, and collective indices are skewsymmetric pairs $\alpha\beta \to a$, $\gamma\delta \to b$. In more detail, if

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then in virtue of (102) for the metric tensor of $\mathbb{R}^6$ we obtain

$$g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{103}$$

where the order of collective indices in $\mathbb{R}^6$ is $23 \to 0$, $10 \to 1$, $20 \to 2$, $30 \to 3$, $31 \to 4$, $12 \to 5$. As it is shown in [56], the Lorentz transformations can be represented by linear transformations of the space $\mathbb{R}^6$. Let us write an invariance condition of the system (101). Let $\mathfrak{g} : a' = \mathfrak{g}^{-1}a$ be a transformation of the bivector space $\mathbb{R}^6$, that is, $a' = \sum_{b=1}^{6} g_{ba}a_b$, where $a = (a_1, a_2, a_3, a_1^*, a_2^*, a_3^*)$ and g_{ba} is the metric tensor (103). We can write the tensor (103) in the form $g_{ab} = \begin{pmatrix} g^-_{ik} & \\ & g^+_{ik} \end{pmatrix}$, then $a' = \sum_{k=1}^{3} g^-_{ki}a_k$, $a^{*\prime} = \sum_{k=1}^{3} g^+_{ki}a^*_k$. Replacing ψ via $T_{\mathfrak{g}}^{-1}\psi'$, and differentiation on a_k (a_k^*) by differentiation on a'_k ($a_k^{*\prime}$) via the formulae

$$\frac{\partial}{\partial a_k} = \sum g^-_{ik}\frac{\partial}{\partial a'_i}, \quad \frac{\partial}{\partial a^*_k} = \sum g^+_{ik}\frac{\partial}{\partial a^{*\prime}_i},$$

we obtain

$$\sum_i \Bigg[g^-_{i1}\Lambda_1 T_{\mathfrak{g}}^{-1}\frac{\partial \psi'}{\partial a'_i} + g^-_{i2}\Lambda_2 T_{\mathfrak{g}}^{-1}\frac{\partial \psi'}{\partial a'_i} + g^-_{i3}\Lambda_3 T_{\mathfrak{g}}^{-1}\frac{\partial \psi'}{\partial a'_i} -$$

$$- i g^-_{i1}\Lambda_1 T_{\mathfrak{g}}^{-1}\frac{\partial \psi'}{\partial a^{*\prime}_i} - i g^-_{i2}\Lambda_2 T_{\mathfrak{g}}^{-1}\frac{\partial \psi'}{\partial a^{*\prime}_i} - i g^-_{i3}\Lambda_3 T_{\mathfrak{g}}^{-1}\frac{\partial \psi'}{\partial a^{*\prime}_i}\Bigg] + \kappa^c T_{\mathfrak{g}}^{-1}\psi' = 0,$$

$$\sum_i \Bigg[g^+_{i1}\Lambda^*_1 \overset{*}{T}{}_{\mathfrak{g}}^{-1}\frac{\partial \dot\psi'}{\partial \widetilde{a}'_i} + g^+_{i2}\Lambda^*_2 \overset{*}{T}{}_{\mathfrak{g}}^{-1}\frac{\partial \dot\psi'}{\partial \widetilde{a}'_i} + g^+_{i3}\Lambda^*_3 \overset{*}{T}{}_{\mathfrak{g}}^{-1}\frac{\partial \dot\psi'}{\partial \widetilde{a}'_i} +$$

$$+ i g^+_{i1}\Lambda^*_1 \overset{*}{T}{}_{\mathfrak{g}}^{-1}\frac{\partial \dot\psi'}{\partial \widetilde{a^*}'_i} + i g^+_{i2}\Lambda^*_2 \overset{*}{T}{}_{\mathfrak{g}}^{-1}\frac{\partial \dot\psi'}{\partial \widetilde{a^*}'_i} + i g^+_{i3}\Lambda^*_3 \overset{*}{T}{}_{\mathfrak{g}}^{-1}\frac{\partial \dot\psi'}{\partial \widetilde{a^*}'_i}\Bigg] + \dot\kappa^c \overset{*}{T}{}_{\mathfrak{g}}^{-1}\dot\psi' = 0.$$

For coincidence of the latter system with (101) we must multiply this system by $T_{\mathfrak{g}}$ ($\overset{*}{T}_{\mathfrak{g}}$) from the left:

$$\begin{aligned}
\sum_i\sum_k g^-_{ik}T_{\mathfrak{g}}\Lambda_k T^{-1}_{\mathfrak{g}}\frac{\partial\psi'}{\partial a'_i}-i\sum_i\sum_k g^-_{ik}T_{\mathfrak{g}}\Lambda_k T^{-1}_{\mathfrak{g}}\frac{\partial\psi'}{\partial a^{*\prime}_i}+\kappa^c\psi' &= 0,\\
\sum_i\sum_k g^+_{ik}\overset{*}{T}_{\mathfrak{g}}\Lambda^*_k\overset{*}{T}{}^{-1}_{\mathfrak{g}}\frac{\partial\dot{\psi}'}{\partial\widetilde{a}'_i}+i\sum_i\sum_k g^+_{ik}\overset{*}{T}_{\mathfrak{g}}\Lambda^*_k\overset{*}{T}{}^{-1}_{\mathfrak{g}}\frac{\partial\dot{\psi}'}{\partial\widetilde{a}^*{}_i'}+\dot{\kappa}^c\dot{\psi}' &= 0.
\end{aligned}$$

The requirement of invariance means that for any transformation $\mathfrak{g}$ between the matrices Λ_k (Λ^*_k) we must have the relations

$$\begin{aligned}
\sum_k g^-_{ik}T_{\mathfrak{g}}\Lambda_k T^{-1}_{\mathfrak{g}} &= \Lambda_i,\\
\sum_k g^+_{ik}\overset{*}{T}_{\mathfrak{g}}\Lambda^*_k\overset{*}{T}{}^{-1}_{\mathfrak{g}} &= \Lambda^*_i,
\end{aligned}\tag{104}$$

where Λ^*_i are the matrices of the equations in the dual representation space, κ^c is a complex number, $\partial/\partial\widetilde{a}_i$ mean covariant derivatives in the dual space.

Let us find commutation relations between the matrices Λ_i, Λ^*_i and infinitesimal operators (45), (46), (47), (48) defined in the helicity basis. First of all, let us present transformations $T_{\mathfrak{g}}$ ($\overset{*}{T}_{\mathfrak{g}}$) in the infinitesimal form, $\mathsf{I}+\mathsf{A}_i\xi+\ldots$, $\mathsf{I}+\mathsf{B}_i\xi+\ldots$, $\mathsf{I}+\widetilde{\mathsf{A}}_i\xi+\ldots$, $\mathsf{I}+\widetilde{\mathsf{B}}_i\xi+\ldots$. Substituting these transformations into invariance conditions (104), we obtain with an accuracy of the terms of second order the following commutation relations

$$\begin{array}{lllllllll}
[\mathsf{A}_1,\Lambda_1] &=& 0, & [\mathsf{A}_1,\Lambda_2] &=& \Lambda_3, & [\mathsf{A}_1,\Lambda_3] &=& -\Lambda_2,\\
[\mathsf{A}_2,\Lambda_1] &=& -\Lambda_3, & [\mathsf{A}_2,\Lambda_2] &=& 0, & [\mathsf{A}_2,\Lambda_3] &=& \Lambda_1,\\
[\mathsf{A}_3,\Lambda_1] &=& \Lambda_2, & [\mathsf{A}_3,\Lambda_2] &=& -\Lambda_1, & [\mathsf{A}_3,\Lambda_3] &=& 0.
\end{array}\tag{105}$$

$$\begin{array}{lllllllll}
[\mathsf{B}_1,\Lambda_1] &=& 0, & [\mathsf{B}_1,\Lambda_2] &=& -i\Lambda_3, & [\mathsf{B}_1,\Lambda_3] &=& i\Lambda_2,\\
[\mathsf{B}_2,\Lambda_1] &=& i\Lambda_3, & [\mathsf{B}_2,\Lambda_2] &=& 0, & [\mathsf{B}_2,\Lambda_3] &=& -i\Lambda_1,\\
[\mathsf{B}_3,\Lambda_1] &=& -i\Lambda_2, & [\mathsf{B}_3,\Lambda_2] &=& i\Lambda_1, & [\mathsf{B}_3,\Lambda_3] &=& 0.
\end{array}\tag{106}$$

$$\begin{array}{lllllllll}
\left[\widetilde{\mathsf{A}}_1,\Lambda^*_1\right] &=& 0, & \left[\widetilde{\mathsf{A}}_1,\Lambda^*_2\right] &=& \Lambda^*_3, & \left[\widetilde{\mathsf{A}}_1,\Lambda^*_3\right] &=& -\Lambda^*_2,\\
\left[\widetilde{\mathsf{A}}_2,\Lambda^*_1\right] &=& -\Lambda^*_3, & \left[\widetilde{\mathsf{A}}_2,\Lambda^*_2\right] &=& 0, & \left[\widetilde{\mathsf{A}}_2,\Lambda^*_3\right] &=& \Lambda^*_1,\\
\left[\widetilde{\mathsf{A}}_3,\Lambda^*_1\right] &=& \Lambda^*_2, & \left[\widetilde{\mathsf{A}}_3,\Lambda^*_2\right] &=& -\Lambda^*_1, & \left[\widetilde{\mathsf{A}}_3,\Lambda^*_3\right] &=& 0.
\end{array}\tag{107}$$

$$\begin{array}{lllllllll}
\left[\widetilde{\mathsf{B}}_1,\Lambda^*_1\right] &=& 0, & \left[\widetilde{\mathsf{B}}_1,\Lambda^*_2\right] &=& i\Lambda^*_3, & \left[\widetilde{\mathsf{B}}_1,\Lambda^*_3\right] &=& -i\Lambda^*_2,\\
\left[\widetilde{\mathsf{B}}_2,\Lambda^*_1\right] &=& -i\Lambda^*_3, & \left[\widetilde{\mathsf{B}}_2,\Lambda^*_2\right] &=& 0, & \left[\widetilde{\mathsf{B}}_2,\Lambda^*_3\right] &=& i\Lambda^*_1,\\
\left[\widetilde{\mathsf{B}}_3,\Lambda^*_1\right] &=& i\Lambda^*_2, & \left[\widetilde{\mathsf{B}}_3,\Lambda^*_2\right] &=& -i\Lambda^*_1, & \left[\widetilde{\mathsf{B}}_3,\Lambda^*_3\right] &=& 0.
\end{array}\tag{108}$$

Further, using the latter relations and taking into account (41), it is easy to establish commutation relations between Λ_3, Λ^*_3 and generators $\mathsf{Y}_\pm$, Y_3, $\mathsf{X}_\pm$, X_3:

$$\begin{array}{ll}
[[\Lambda_3,\mathsf{X}_-],\mathsf{X}_+]=2\Lambda_3, & [[\Lambda^*_3,\mathsf{Y}_-],\mathsf{Y}_+]=2\Lambda^*_3,\\
[\Lambda_3,\mathsf{X}_3]=0, & [\Lambda^*_3,\mathsf{Y}_3]=0,
\end{array}\tag{109}$$

Using the relations (109), we will find an explicit form of the matrices Λ_3 and Λ^*_3, and after this we will find Λ_1, Λ_2 and Λ^*_1, Λ^*_2. The wave function $\boldsymbol{\psi}$ is transformed within some representation $T_{\mathfrak{g}}$ of the group $SL(2,\mathbb{C})$. We assume that $T_{\mathfrak{g}}$ is decomposed into irreducible representations. The components of the function $\boldsymbol{\psi}$ we will numerate by the indices l and m, where l is a weight of irreducible representation, m is a number of the components in the representation of the weight l. In the case when a representation with one and the same weight l at the decomposition of $\boldsymbol{\psi}$ occurs more than one time, then with the aim to distinguish these representations we will add the index k, which indicates a number of the representations of the weight l. Denoting $\zeta_{lm;\dot{l}\dot{m}} =\mid lm;\dot{l}\dot{m}\rangle$ and coming to the helicity basis, we obtain a following decomposition for the wave function:

$$\boldsymbol{\psi}(a_1,a_2,a_3,a^*_1,a^*_2,a^*_3)=\sum_{l,m,k}\psi^k_{lm;\dot{l}\dot{m}}(a_1,a_2,a_3,a^*_1,a^*_2,a^*_3)\zeta^k_{lm;\dot{l};\dot{m}},$$

where $a_1,a_2,a_3,a^*_1,a^*_2,a^*_3$ are the coordinates of the complex space $\mathbb{C}^3\simeq\mathbb{R}^6$ (parameters of $SL(2,\mathbb{C})$)[2]. Analogously, for the dual representation we have

$$\dot{\boldsymbol{\psi}}(\widetilde{a}_1,\widetilde{a}_2,\widetilde{a}_3,\widetilde{a^*}_1,\widetilde{a^*}_2,\widetilde{a^*}_3)=\sum_{\dot{l},\dot{m},\dot{k}}\psi^{\dot{k}}_{\dot{l}\dot{m};lm}(\widetilde{a}_1,\widetilde{a}_2,\widetilde{a}_3,\widetilde{a^*}_1,\widetilde{a^*}_2,\widetilde{a^*}_3)\zeta^{\dot{k}}_{\dot{l}\dot{m};lm}.$$

The transformation Λ_3 in the helicity basis has a form

$$\Lambda_3\zeta^k_{lm;\dot{l}\dot{m}}=\sum_{l',m',k'}c^{k'k}_{l'l,m'm}\zeta^{k'}_{l'm';\dot{l}\dot{m}}.$$

Calculating the commutators $[\Lambda_3,\mathsf{X}_3]$, $[[\Lambda_3,\mathsf{X}_-],\mathsf{X}_+]$, we find the numbers $c^{k'k}_{l'l,m'm}$:

$$\Lambda_3:\quad\left\{\begin{array}{rcl}
c^{k'k}_{l-1,l,m} &=& c^{k'k}_{l-1,l}\sqrt{l^2-m^2},\\
c^{k'k}_{l,l,m} &=& c^{k'k}_{ll}m,\\
c^{k'k}_{l+1,l,m} &=& c^{k'k}_{l+1,l}\sqrt{(l+1)^2-m^2}.
\end{array}\right.\tag{110}$$

All other elements of the matrix Λ_3 are equal to zero. Let us define now elements of the matrices Λ_1 and Λ_2. For the transformations Λ_1 and Λ_2 in the helicity basis we have

$$\begin{aligned}
\Lambda_1\zeta^k_{lm;\dot{l}\dot{m}} &= \sum_{l',m',k'}a^{k'k}_{l'l,m'm}\zeta^{k'}_{l'm';\dot{l}\dot{m}},\\
\Lambda_2\zeta^k_{lm;\dot{l}\dot{m}} &= \sum_{l',m',k'}b^{k'k}_{l'l,m'm}\zeta^{k'}_{l'm';\dot{l}\dot{m}}.
\end{aligned}$$

Using the relations $\Lambda_1=[\mathsf{A}_2,\Lambda_3]$ (or $\Lambda_1=i\,[\mathsf{B}_2,\Lambda_3]$) and (45) (or (46)), and also (110), we find the elements $a^{k'k}_{l'l,m'm}$ of the matrix Λ_1. Analogously, from the relations $\Lambda_2=-\,[\mathsf{A}_1,\Lambda_3]$ (or $\Lambda_2=-i\,[\mathsf{B}_1,\Lambda_3]$) and (45) (or (46)), (110) we obtain the elements $b^{k'k}_{l'l,m'm}$ of Λ_2. Thus,

$$\Lambda_1:\quad\left\{\begin{array}{rcl}
a^{k'k}_{l-1,l,m-1,m} &=& -\frac{c_{l-1,l}}{2}\sqrt{(l+m)(l+m-1)},\\
a^{k'k}_{l,l,m-1,m} &=& \frac{c_{ll}}{2}\sqrt{(l+m)(l-m+1)},\\
a^{k'k}_{l+1,l,m-1,m} &=& \frac{c_{l+1,l}}{2}\sqrt{(l-m+1)(l-m+2)},\\
a^{k'k}_{l-1,l,m+1,m} &=& \frac{c_{l-1,l}}{2}\sqrt{(l-m)(l-m-1)},\\
a^{k'k}_{l,l,m+1,m} &=& \frac{c_{ll}}{2}\sqrt{(l+m+1)(l-m)},\\
a^{k'k}_{l+1,l,m+1,m} &=& -\frac{c_{l+1,l}}{2}\sqrt{(l+m+1)(l+m+2)}.
\end{array}\right.\tag{111}$$

[2]Recall that the wave function $\boldsymbol{\psi}(a_j,a^*_j)$ is defined on the group manifold $\mathfrak{L}_6$, that is, $\boldsymbol{\psi}$ is a function on the Lorentz group.

$$
\Lambda_2:\quad\left\{\begin{array}{rcl}
b^{k'k}_{l-1,l,m-1,m} &=& -\frac{ic_{l-1,l}}{2}\sqrt{(l+m)(l+m-1)},\\
b^{k'k}_{l,l,m-1,m} &=& \frac{ic_{ll}}{2}\sqrt{(l+m)(l-m+1)},\\
b^{k'k}_{l+1,l,m-1,m} &=& \frac{ic_{l+1,l}}{2}\sqrt{(l-m+1)(l-m+2)},\\
b^{k'k}_{l-1,l,m+1,m} &=& -\frac{ic_{l-1,l}}{2}\sqrt{(l-m)(l-m-1)},\\
b^{k'k}_{l,l,m+1,m} &=& -\frac{ic_{ll}}{2}\sqrt{(l+m+1)(l-m)},\\
b^{k'k}_{l+1,l,m+1,m} &=& \frac{ic_{l+1,l}}{2}\sqrt{(l+m+1)(l+m+2)}.
\end{array}\right. \tag{112}
$$

Coming to the dual representations, we find elements of the matrices Λ^*_1, Λ^*_2 and Λ^*_3. The transformations Λ^*_i in the dual helicity basis are

$$
\begin{array}{rcl}
\Lambda^*_1\zeta^{\dot{k}}_{\dot{l}\dot{m};lm} &=& \sum\limits_{\dot{l}',\dot{m}',\dot{k}'} d^{\dot{k}'\dot{k}}_{\dot{l}'\dot{l},\dot{m}'\dot{m}}\zeta^{\dot{k}'}_{\dot{l}'\dot{m}';lm},\\
\Lambda^*_2\zeta^{\dot{k}}_{\dot{l}\dot{m};lm} &=& \sum\limits_{\dot{l}',\dot{m}',\dot{k}'} e^{\dot{k}'\dot{k}}_{\dot{l}'\dot{l},\dot{m}'\dot{m}}\zeta^{\dot{k}'}_{\dot{l}'\dot{m}';lm},\\
\Lambda^*_3\zeta^{\dot{k}}_{\dot{l}\dot{m};lm} &=& \sum\limits_{\dot{l}',\dot{m}',\dot{k}'} f^{\dot{k}'\dot{k}}_{\dot{l}'\dot{l},\dot{m}'\dot{m}}\zeta^{\dot{k}'}_{\dot{l}'\dot{m}';lm}.
\end{array}
$$

Calculating the commutators $[\Lambda^*_3,\mathsf{Y}_3]$, $[[\Lambda^*_3,\mathsf{Y}_-],\mathsf{Y}_+]$ with respect to the vectors $\zeta^{\dot{k}}_{\dot{l}\dot{m};lm}$ of the dual basis, we find elements of the matrix Λ^*_3. Using the relations $\Lambda^*_1=\left[\widetilde{\mathsf{A}}_2,\Lambda^*_3\right]$ (or $\Lambda^*_1=-i\left[\widetilde{\mathsf{B}}_2,\Lambda^*_3\right]$) and (47) (or (48)), we find elements $d^{\dot{k}'\dot{k}}_{\dot{l}'\dot{l},\dot{m}'\dot{m}}$ of the matrix Λ^*_1. And also from the relations $\Lambda^*_2=-\left[\widetilde{\mathsf{A}}_1,\Lambda^*_3\right]$ (or $\Lambda^*_2=i\left[\widetilde{\mathsf{B}}_1,\Lambda^*_3\right]$) we obtain elements $e^{\dot{k}'\dot{k}}_{\dot{l}'\dot{l},\dot{m}'\dot{m}}$ of Λ^*_2. All calculations are analogous to the calculations presented for the case of Λ_i. In the result we have

$$
\Lambda^*_1:\quad\left\{\begin{array}{rcl}
d^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}-1,\dot{m}} &=& -\frac{c_{\dot{l}-1,\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}-1)},\\
d^{\dot{k}'\dot{k}}_{\dot{l},\dot{l},\dot{m}-1,\dot{m}} &=& \frac{c_{\dot{l}\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}+1)},\\
d^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l},\dot{m}-1,\dot{m}} &=& \frac{c_{\dot{l}+1,\dot{l}}}{2}\sqrt{(\dot{l}-\dot{m}+1)(\dot{l}-\dot{m}+2)},\\
d^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}+1,\dot{m}} &=& \frac{c_{\dot{l}-1,\dot{l}}}{2}\sqrt{(\dot{l}-\dot{m})(\dot{l}-\dot{m}-1)},\\
d^{\dot{k}'\dot{k}}_{\dot{l},\dot{l},\dot{m}+1,\dot{m}} &=& \frac{c_{\dot{l}\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}-\dot{m})},\\
d^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l},\dot{m}+1,\dot{m}} &=& -\frac{c_{\dot{l}+1,\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}+\dot{m}+2)}.
\end{array}\right. \tag{113}
$$

$$
\Lambda^*_2:\quad\left\{\begin{array}{rcl}
e^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}-1,\dot{m}} &=& -\frac{ic_{\dot{l}-1,\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}-1)},\\
e^{\dot{k}'\dot{k}}_{\dot{l},\dot{l},\dot{m}-1,\dot{m}} &=& \frac{ic_{\dot{l}\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}+1)},\\
e^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l},\dot{m}-1,\dot{m}} &=& \frac{ic_{\dot{l}+1,\dot{l}}}{2}\sqrt{(\dot{l}-\dot{m}+1)(\dot{l}-\dot{m}+2)},\\
e^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}+1,\dot{m}} &=& \frac{-ic_{\dot{l}-1,\dot{l}}}{2}\sqrt{(\dot{l}-\dot{m})(\dot{l}-\dot{m}-1)},\\
e^{\dot{k}'\dot{k}}_{\dot{l},\dot{l},\dot{m}+1,\dot{m}} &=& \frac{-ic_{\dot{l}\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}-\dot{m})},\\
e^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l},\dot{m}+1,\dot{m}} &=& -\frac{ic_{\dot{l}+1,\dot{l}}}{2}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}+\dot{m}+2)}.
\end{array}\right. \tag{114}
$$

$$
\Lambda_3^*: \quad \begin{cases} f^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l},\dot{m}} &= c^{\dot{k}'\dot{k}}_{\dot{l}-1,\dot{l}}\sqrt{\dot{l}^2-\dot{m}^2}, \\ f^{\dot{k}'\dot{k}}_{\dot{l}\dot{l},\dot{m}} &= c^{\dot{k}'\dot{k}}_{\dot{l}\dot{l}}\dot{m}, \\ f^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l},\dot{m}} &= c^{\dot{k}'\dot{k}}_{\dot{l}+1,\dot{l}}\sqrt{(\dot{l}+1)^2-\dot{m}^2}. \end{cases} \tag{115}
$$

7.1 Boundary value problem

Following to the classical methods of mathematical physics [20], it is quite natural to set up *a boundary value problem for the relativistic wave equation (relativistically invariant system).* It is well known that all the physically meaningful requirements, which follow from the experience, are contained in the boundary value problem.

We will set up a boundary value problem for the two-dimensional complex sphere $\mathbb{S}^2$ (this problem can be considered as a relativistic generalization of the classical Dirichlet problem for the sphere S^2).

Let T be an unbounded region in $\mathbb{C}^3 \simeq \mathbb{R}^6$ and let Σ be a surface of the complex two-sphere (correspondingly, $\dot{\Sigma}$, for the dual two-sphere), then it needs to find a function $\boldsymbol{\psi}(\mathfrak{g}) = (\psi_m(\mathfrak{g}), \dot{\psi}_{\dot{m}}(\mathfrak{g}))^T$ satisfying the following conditions:

1) $\boldsymbol{\psi}(\mathfrak{g})$ is a solution of the system

$$
\sum_{j=1}^{3}\Lambda_j\frac{\partial\psi}{\partial a_j} - i\sum_{j=1}^{3}\Lambda_j\frac{\partial\psi}{\partial a_j^*} + \dot{\kappa}^c\dot{\psi} = 0, \tag{116}
$$

$$
\sum_{j=1}^{3}\Lambda_j^*\frac{\partial\dot{\psi}}{\partial\widetilde{a}_j} + i\sum_{j=1}^{3}\Lambda_j^*\frac{\partial\dot{\psi}}{\partial\widetilde{a}_j^*} + \kappa^c\psi = 0, \tag{117}
$$

in the all region T;

2) $\boldsymbol{\psi}(\mathfrak{g})$ is a continuous function (everywhere in T), including the surfaces Σ and $\dot{\Sigma}$;

3) $\psi_m(\mathfrak{g})\big|_\Sigma = F_m(\mathfrak{g})$, $\dot{\psi}_{\dot{m}}(\mathfrak{g})\Big|_{\dot{\Sigma}} = \dot{F}_{\dot{m}}(\mathfrak{g})$, where $F_m(\mathfrak{g})$ and $\dot{F}_{\dot{m}}(\mathfrak{g})$ are square integrable functions defined on the surfaces Σ and $\dot{\Sigma}$, respectively.

In particular, boundary conditions can be represented by constants,

$$
\psi(\mathfrak{g})\Big|_\Sigma = \text{const} = F_0, \qquad \dot{\psi}(\mathfrak{g})\Big|_{\dot{\Sigma}} = \text{const} = \dot{F}_0.
$$

It is obvious that an explicit form of the boundary conditions follows from the experience. For example, they can describe a distribution of energy in the experiment.

With the aim to solve the boundary value problem we come to the complex Euler angles (36) and represent the function $\boldsymbol{\psi}(r,\theta^c,\varphi^c) = (\psi_m(r,\theta^c,\varphi^c), \dot{\psi}_{\dot{m}}(r^*,\dot{\theta}^c,\dot{\varphi}^c))^T$ in the form of following series

$$
\psi_m(r,\theta^c,\varphi^c) = \sum_{l=0}^{\infty}\sum_k \boldsymbol{f}_{lmk}(r)\sum_{n=-l}^{l}\alpha^m_{l,n}\mathfrak{M}^l_{mn}(\varphi,\epsilon,\theta,\tau,0,0), \tag{118}
$$

$$
\dot{\psi}_{\dot{m}}(r^*,\dot{\theta}^c,\dot{\varphi}^c) = \sum_{\dot{l}=0}^{\infty}\sum_{\dot{k}} \boldsymbol{f}_{\dot{l}\dot{m}\dot{k}}(r^*)\sum_{\dot{n}=-\dot{l}}^{\dot{l}}\alpha^{\dot{m}}_{\dot{l},\dot{n}}\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0), \tag{119}
$$

where

$$\alpha^m_{l,n} = \frac{(-1)^n(2l+1)(2\dot{l}+1)}{32\pi^4}\int\limits_{\mathbb{S}^2} F_m(\theta^c,\varphi^c)\mathfrak{M}^n_l(\varphi,\epsilon,\theta,\tau,0,0)\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\tau d\epsilon,$$

$$\alpha^{\dot{m}}_{\dot{l},\dot{n}} = \frac{(-1)^{\dot{n}}(2l+1)(2\dot{l}+1)}{32\pi^4}\int\limits_{\mathbb{S}^2} \dot{F}_{\dot{m}}(\dot{\theta}^c,\dot{\varphi}^c)\mathfrak{M}^{\dot{n}}_{\dot{l}}(\varphi,\epsilon,\theta,\tau,0,0)\sin\theta^c\sin\dot{\theta}^c d\theta d\varphi d\tau d\epsilon,$$

The index k numerates equivalent representations. $\mathfrak{M}^n_l(\varphi,\epsilon,\theta,\tau,0,0)$ $(\mathfrak{M}^{\dot{n}}_{\dot{l}}(\varphi,\epsilon,\theta,\tau,0,0))$ are associated hyperspherical functions defined on the surface Σ $(\dot{\Sigma})$ of the two-dimensional complex sphere of the radius r (r^*), $\boldsymbol{f}_{lmk}(r)$ and $\boldsymbol{f}_{\dot{l}\dot{m}\dot{k}}(r^*)$ are radial functions. It is easy to see that we come here to the harmonic analysis on the complex two-sphere, since the series (118) and (119) have the structure of the Fourier series on $\mathbb{S}^2$.

General solutions of the system (101) have been found in the work [115] on the tangent bundle $T\mathfrak{L}_6 = \bigcup\limits_{\mathfrak{g}\in\mathfrak{L}_6} T_{\mathfrak{g}}\mathfrak{L}_6$ of the group manifold $\mathfrak{L}_6$. A separation of variables in (101) is realized via the following factorization

$$\begin{aligned}\psi^k_{lm;\dot{l}\dot{m}} &= \boldsymbol{f}^{l_0}_{lmk}(r)\mathfrak{M}^{l_0}_{mn}(\varphi,\epsilon,\theta,\tau,0,0),\\ \psi^{\dot{k}}_{\dot{l}\dot{m};lm} &= \boldsymbol{f}^{\dot{l}_0}_{\dot{l}\dot{m}\dot{k}}(r^*)\mathfrak{M}^{\dot{l}_0}_{\dot{m}\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),\end{aligned} \tag{120}$$

where $l_0 \geq l$, $-l_0 \leq m$, $n \leq l_0$ and $\dot{l}_0 \geq \dot{l}$, $-\dot{l}_0 \leq \dot{m}$, $\dot{n} \leq \dot{l}_0$. In the result of separation of variables the relativistically invariant system (101) is reduced to a system of ordinary differential equations (for more details see [115]):

$$\begin{aligned}
&\sum_{k'} c^{kk'}_{l,l-1}\left[2\sqrt{l^2-m^2}\frac{d\boldsymbol{f}^{l_0}_{l-1,m,k'}(r)}{dr} - \frac{1}{r}(l+1)\sqrt{l^2-m^2}\boldsymbol{f}^{l_0}_{l-1,m,k'}(r)+\right.\\
&\qquad +\frac{1}{r}\sqrt{(l+m)(l+m-1)}\sqrt{(l_0+m)(l_0-m+1)}\boldsymbol{f}^{l_0}_{l-1,m-1,k'}(r)+\\
&\qquad\qquad \left.+\frac{1}{r}\sqrt{(l-m)(l-m-1)}\sqrt{(l_0+m+1)(l_0-m)}\boldsymbol{f}^{l_0}_{l-1,m+1,k'}(r)\right]+\\
&\sum_{k'} c^{kk'}_{ll}\left[2m\frac{d\boldsymbol{f}^{l_0}_{l,m,k'}(r)}{dr} - \frac{1}{r}m\boldsymbol{f}^{l_0}_{l,m,k'}(r)-\right.\\
&\qquad -\frac{1}{r}\sqrt{(l+m)(l-m+1)}\sqrt{(l_0+m)(l_0-m+1)}\boldsymbol{f}^{l_0}_{l,m-1,k'}(r)+\\
&\qquad\qquad \left.+\frac{1}{r}\sqrt{(l+m+1)(l-m)}\sqrt{(l_0+m+1)(l_0-m)}\boldsymbol{f}^{l_0}_{l,m+1,k'}(r)\right]+\\
&\sum_{k'} c^{kk'}_{l,l+1}\left[2\sqrt{(l+1)^2-m^2}\frac{d\boldsymbol{f}^{l_0}_{l+1,m,k'}(r)}{dr} + \frac{1}{r}l\sqrt{(l+1)^2-m^2}\boldsymbol{f}^{l_0}_{l+1,m,k'}(r)-\right.\\
&\qquad -\frac{1}{r}\sqrt{(l-m+1)(l-m+2)}\sqrt{(l_0+m)(l_0-m+1)}\boldsymbol{f}^{l_0}_{l+1,m-1,k'}(r)-\\
&\qquad \left.-\frac{1}{r}\sqrt{(l+m+1)(l+m+2)}\sqrt{(l_0+m+1)(l_0-m)}\boldsymbol{f}^{l_0}_{l+1,m+1,k'}(r)\right]+\\
&\qquad\qquad\qquad + \kappa^c\boldsymbol{f}^{l_0}_{lmk}(r) = 0,
\end{aligned}$$

$$\sum_{\dot{k}'} c^{\dot{k}\dot{k}'}_{\dot{l},\dot{l}-1}\left[2\sqrt{\dot{l}^2-\dot{m}^2}\frac{d\boldsymbol{f}^{\dot{l}_0}_{\dot{l}-1,\dot{m},\dot{k}'}(r^*)}{dr^*}-\frac{1}{r^*}(\dot{l}+1)\sqrt{\dot{l}^2-\dot{m}^2}\boldsymbol{f}^{\dot{l}_0}_{\dot{l}-1,\dot{m},\dot{k}'}(r^*)+\right.$$

$$+\frac{1}{r^*}\sqrt{(\dot{l}+\dot{m})(\dot{l}+\dot{m}-1)}\sqrt{(\dot{l}_0+\dot{m})(\dot{l}_0-\dot{m}+1)}\boldsymbol{f}^{\dot{l}_0}_{\dot{l}-1,\dot{m}-1,\dot{k}'}(r^*)+$$

$$\left.+\frac{1}{r^*}\sqrt{(\dot{l}-\dot{m})(\dot{l}-\dot{m}-1)}\sqrt{(\dot{l}_0+\dot{m}+1)(\dot{l}_0-\dot{m})}\boldsymbol{f}^{\dot{l}_0}_{\dot{l}-1,\dot{m}+1,\dot{k}'}(r^*)\right]+$$

$$\sum_{\dot{k}'} c^{\dot{k}\dot{k}'}_{\dot{l},\dot{l}}\left[2\dot{m}\frac{d\boldsymbol{f}^{\dot{l}_0}_{\dot{l},\dot{m},\dot{k}'}(r^*)}{dr^*}-\frac{1}{r^*}\dot{m}\boldsymbol{f}^{\dot{l}_0}_{\dot{l},\dot{m},\dot{k}'}(r^*)-\right.$$

$$-\frac{1}{r^*}\sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}-1)}\sqrt{(\dot{l}_0+\dot{m})(\dot{l}_0-\dot{m}+1)}\boldsymbol{f}^{\dot{l}_0}_{\dot{l},\dot{m}-1,\dot{k}'}(r^*)+$$

$$\left.+\frac{1}{r^*}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}-\dot{m})}\sqrt{(\dot{l}_0+\dot{m}+1)(\dot{l}_0-\dot{m})}\boldsymbol{f}^{\dot{l}_0}_{\dot{l},\dot{m}+1,\dot{k}'}(r^*)\right]+$$

$$\sum_{\dot{k}'} c^{\dot{k}\dot{k}'}_{\dot{l},\dot{l}+1}\left[2\sqrt{(\dot{l}+1)^2-\dot{m}^2}\frac{d\boldsymbol{f}^{\dot{l}_0}_{\dot{l}+1,\dot{m},\dot{k}'}(r^*)}{dr^*}+\frac{1}{r^*}\dot{l}\sqrt{(\dot{l}+1)^2-\dot{m}^2}\boldsymbol{f}^{\dot{l}_0}_{\dot{l}+1,\dot{m},\dot{k}'}(r^*)-\right.$$

$$-\frac{1}{r^*}\sqrt{(\dot{l}-\dot{m}+1)(\dot{l}-\dot{m}+2)}\sqrt{(\dot{l}_0+\dot{m})(\dot{l}_0-\dot{m}+1)}\boldsymbol{f}^{\dot{l}_0}_{\dot{l}+1,\dot{m}-1,\dot{k}'}(r^*)-$$

$$\left.-\frac{1}{r^*}\sqrt{(\dot{l}+\dot{m}+1)(\dot{l}+\dot{m}+2)}\sqrt{(\dot{l}_0+\dot{m}+1)(\dot{l}_0-\dot{m})}\boldsymbol{f}^{\dot{l}_0}_{\dot{l}+1,\dot{m}+1,\dot{k}'}(r^*)\right]+$$

$$+\dot{\kappa}^c\boldsymbol{f}^{\dot{l}_0}_{\dot{l}\dot{m}\dot{k}'}(r^*)=0. \quad (121)$$

Substituting solutions of this system into the series (118) and (119), we obtain a solution of the boundary value problem. It is easy to see that a boundary value problem of the same type can be defined on the homogeneous spaces $\mathcal{M}_8$ and $\mathcal{M}_6$. The analogous problem on the homogeneous space $\mathcal{M}_7$ leads to relativistic wave equations in 2+1 dimensions [53, 85, 19, 41] (or relativistically invariant system on the group $SU(1,1)$). The boundary value problem for the hyperboloid $\mathbb{H}^3$ comes beyond the framework of this paper and will be studied in a future work.

8 The Dirac field

In this section we will consider a boundary value problem for the Dirac field $(1/2,0)\oplus(0,1/2)$ (electron-positron field) defined on the homogeneous space $\mathcal{M}_8$. Solution of this problem allows us to construct field operators and further to define a quantization procedure for the Dirac field on the space $\mathcal{M}_8$.

We start with the Lagrangian (98) on the group manifold $\mathcal{M}_{10}$:

$$\mathcal{L}(\boldsymbol{\alpha})=-\frac{1}{2}\left(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\Gamma_\mu\frac{\partial\boldsymbol{\psi}(\boldsymbol{\alpha})}{\partial x_\mu}-\frac{\partial\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})}{\partial x_\mu}\Gamma_\mu\boldsymbol{\psi}(\boldsymbol{\alpha})\right)-$$

$$-\frac{1}{2}\left(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\Upsilon_\nu\frac{\partial\boldsymbol{\psi}(\boldsymbol{\alpha})}{\partial\mathfrak{g}_\nu}-\frac{\partial\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})}{\partial\mathfrak{g}_\nu}\Upsilon_\nu\boldsymbol{\psi}(\boldsymbol{\alpha})\right)-\kappa\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}), \quad (122)$$

where $\boldsymbol{\psi}(\boldsymbol{\alpha})=\psi(x)\psi(\mathfrak{g})$ ($\mu=0,1,2,3,\ \nu=1,\ldots,6$), and

$$\gamma_0=\begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix},\ \gamma_1=\begin{pmatrix}0 & \sigma_1\\ -\sigma_1 & 0\end{pmatrix},\ \gamma_2=\begin{pmatrix}0 & \sigma_2\\ -\sigma_2 & 0\end{pmatrix},\ \gamma_3=\begin{pmatrix}0 & \sigma_3\\ -\sigma_3 & 0\end{pmatrix}, \tag{123}$$

$$\Upsilon_1=\begin{pmatrix}0 & \Lambda^\ast_1\\ \Lambda_1 & 0\end{pmatrix},\quad \Upsilon_2=\begin{pmatrix}0 & \Lambda^\ast_2\\ \Lambda_2 & 0\end{pmatrix},\quad \Upsilon_3=\begin{pmatrix}0 & \Lambda^\ast_3\\ \Lambda_3 & 0\end{pmatrix}, \tag{124}$$

$$\Upsilon_4=\begin{pmatrix}0 & i\Lambda^\ast_1\\ i\Lambda_1 & 0\end{pmatrix},\quad \Upsilon_5=\begin{pmatrix}0 & i\Lambda^\ast_2\\ i\Lambda_2 & 0\end{pmatrix},\quad \Upsilon_6=\begin{pmatrix}0 & i\Lambda^\ast_3\\ i\Lambda_3 & 0\end{pmatrix}, \tag{125}$$

where σ_i are the Pauli matrices, and the matrices Λ_j and $\Lambda^\ast_j$ are derived from (111)–(110) and (113)–(115) at $l=1/2$:

$$\begin{array}{c}
\Lambda_1=\frac{1}{2}c_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \Lambda_2=\frac{1}{2}c_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \Lambda_3=\frac{1}{2}c_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\\
\Lambda^\ast_1=\frac{1}{2}\dot{c}_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \Lambda^\ast_2=\frac{1}{2}\dot{c}_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \Lambda^\ast_3=\frac{1}{2}\dot{c}_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.
\end{array} \tag{126}$$

It is easy to see that these matrices coincide with the Pauli matrices σ_i when $c_{\frac{1}{2}\frac{1}{2}}=2$.

Varying independently $\psi(x)$ and $\overline{\psi}(x)$ in the Lagrangian (122), and then $\psi(\mathfrak{g})$ and $\overline{\psi}(\mathfrak{g})$, we come to the following equations:

$$\left.\begin{array}{rcl}
\gamma_i\dfrac{\partial\psi(x)}{\partial x_i}+\kappa\psi(x) &=& 0,\\
\gamma^T_i\dfrac{\partial\overline{\psi}(x)}{\partial x_i}-\kappa\overline{\psi}(x) &=& 0.
\end{array}\right.\quad (i=1,\ldots,4) \tag{127}$$

$$\left.\begin{array}{rcl}
\Upsilon_k\dfrac{\partial\psi(\mathfrak{g})}{\partial\mathfrak{g}_k}+\kappa\psi(\mathfrak{g}) &=& 0,\\
\Upsilon^T_k\dfrac{\overline{\psi}(\mathfrak{g})}{\partial\mathfrak{g}_k}-\kappa\overline{\psi}(\mathfrak{g}) &=& 0,
\end{array}\right.\quad (k=1,\ldots,6) \tag{128}$$

Now we can formulate the boundary value problem. *Let T be an unbounded region in $\mathcal{M}_8=\mathbb{R}^{1,3}\times\mathbb{S}^2$ and let Σ ($\dot{\Sigma}$) be a surface of the complex two-sphere, then it needs to find the function $\boldsymbol{\psi}(\boldsymbol{\alpha})=(\psi_1(\boldsymbol{\alpha}),\psi_2(\boldsymbol{\alpha}),\dot{\psi}_1(\boldsymbol{\alpha}),\dot{\psi}_2(\boldsymbol{\alpha}))^T$, such that*
1) $\boldsymbol{\psi}(\boldsymbol{\alpha})$ satisfies the equations (127) and (128) in the all region T.
2) $\boldsymbol{\psi}(\boldsymbol{\alpha})$ is a continuous function everywhere in T.
3) $\left.\psi_m(\boldsymbol{\alpha})\right|_{\Sigma}=F_m(\boldsymbol{\alpha})$, $\left.\dot{\psi}_m(\boldsymbol{\alpha})\right|_{\dot{\Sigma}}=\dot{F}_m(\boldsymbol{\alpha})$, where $F_m(\boldsymbol{\alpha})$ and $\dot{F}_m(\boldsymbol{\alpha})$ are square integrable and infinitely differentiable finctions in $\mathcal{M}_8$, $m=1,2$.

The first equation from (127) coincides with the Dirac equation, and the second equation coincides with the Dirac equation for antiparticle. As is known, solutions of these equations are found in the plane-wave approximation, that is, in the form [14, 93][3]:

$$\begin{array}{rcl}
\psi^+(x) &=& u(\mathbf{p})e^{-ipx},\\
\psi^-(x) &=& v(\mathbf{p})e^{ipx},
\end{array}$$

[3]This form is a direct consequence of the T_4-structure of the field equations (127), since the variables x_i are parameters of T_4 and all the irreducible representations of T_4 are expressed via the exponentials.

where the solutions $\psi^+(x)$ and $\psi^-(x)$ correspond to positive and negative energy, respectively, and the amplitudes $u(\mathbf{p})$ and $v(\mathbf{p})$ have the following components

$$u_1(\mathbf{p}) = \left(\frac{E+m}{2m}\right)^{1/2}\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_+}{E+m}\end{pmatrix},\quad u_2(\mathbf{p}) = \left(\frac{E+m}{2m}\right)^{1/2}\begin{pmatrix}0\\1\\\frac{p_-}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix},$$
$$v_1(\mathbf{p}) = \left(\frac{E+m}{2m}\right)^{1/2}\begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_+}{E+m}\\1\\0\end{pmatrix},\quad v_2(\mathbf{p}) = \left(\frac{E+m}{2m}\right)^{1/2}\begin{pmatrix}\frac{p_-}{E+m}\\\frac{-p_z}{E+m}\\0\\1\end{pmatrix},$$

where $p_\pm = p_x \pm ip_y$.

Further, the first equation from (128) can be written as

$$\begin{aligned}\sum_{j=1}^{3}\Lambda_j\frac{\partial\psi}{\partial a_j} - i\sum_{j=1}^{3}\Lambda_j\frac{\partial\psi}{\partial a_j^*} + \dot{\kappa}^c\dot{\psi} &= 0, &(129)\\ \sum_{j=1}^{3}\Lambda_j^*\frac{\partial\dot{\psi}}{\partial\widetilde{a}_j} + i\sum_{j=1}^{3}\Lambda_j^*\frac{\partial\dot{\psi}}{\partial\widetilde{a}_j^*} + \kappa^c\psi &= 0, &(130)\end{aligned}$$

Or, taking into account the explicit form of the matrices Λ_j (Λ_j^*) given by (126), we obtain

$$\begin{aligned}&-\frac{1}{2}\frac{\partial\dot{\psi}_2}{\partial\widetilde{a}_1} + \frac{i}{2}\frac{\partial\dot{\psi}_2}{\partial\widetilde{a}_2} - \frac{1}{2}\frac{\partial\dot{\psi}_1}{\partial\widetilde{a}_3} - \frac{i}{2}\frac{\partial\dot{\psi}_2}{\partial\widetilde{a}_1^*} - \frac{1}{2}\frac{\partial\dot{\psi}_2}{\partial\widetilde{a}_2^*} - \frac{i}{2}\frac{\partial\dot{\psi}_1}{\partial\widetilde{a}_3^*} - \kappa^c\psi_1 = 0,\\ &-\frac{1}{2}\frac{\partial\dot{\psi}_1}{\partial\widetilde{a}_1} - \frac{i}{2}\frac{\partial\dot{\psi}_1}{\partial\widetilde{a}_2} + \frac{1}{2}\frac{\partial\dot{\psi}_2}{\partial\widetilde{a}_3} - \frac{i}{2}\frac{\partial\dot{\psi}_1}{\partial\widetilde{a}_1^*} + \frac{1}{2}\frac{\partial\dot{\psi}_1}{\partial\widetilde{a}_2^*} + \frac{i}{2}\frac{\partial\dot{\psi}_2}{\partial\widetilde{a}_3^*} - \kappa^c\psi_2 = 0,\\ &\frac{1}{2}\frac{\partial\psi_2}{\partial a_1} - \frac{i}{2}\frac{\partial\psi_2}{\partial a_2} + \frac{1}{2}\frac{\partial\psi_1}{\partial a_3} - \frac{i}{2}\frac{\partial\psi_2}{\partial a_1^*} - \frac{1}{2}\frac{\partial\psi_2}{\partial a_2^*} - \frac{i}{2}\frac{\partial\psi_1}{\partial a_3^*} - \dot{\kappa}^c\dot{\psi}_1 = 0,\\ &\frac{1}{2}\frac{\partial\psi_1}{\partial a_1} + \frac{i}{2}\frac{\partial\psi_1}{\partial a_2} - \frac{1}{2}\frac{\partial\psi_2}{\partial a_3} - \frac{i}{2}\frac{\partial\psi_1}{\partial a_1^*} + \frac{1}{2}\frac{\partial\psi_1}{\partial a_2^*} + \frac{i}{2}\frac{\partial\psi_2}{\partial a_3^*} - \dot{\kappa}^c\dot{\psi}_2 = 0, \qquad (131)\end{aligned}$$

The latter system acts on the tangent bundle $T\mathfrak{L}_6$ of the group manifold $\mathfrak{L}_6$. Coming to the helicity basis, we will find solutions of the system (131), that is, we will present components of the Dirac bispinor $\psi = (\psi_1, \psi_2, \dot{\psi}_1, \dot{\psi}_2)^T$ in terms of the functions on the two-dimensional complex sphere (the indices k and $\dot{k}$ we can omit, since representations $\boldsymbol{\tau}_{\frac{1}{2},0}$ and $\boldsymbol{\tau}_{0,\frac{1}{2}}$ occur only one time):

$$\begin{aligned}\psi_1 &= \psi_{\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}} = \boldsymbol{f}^l_{\frac{1}{2},\frac{1}{2}}(r)\mathfrak{M}^l_{\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0),\\ \psi_2 &= \psi_{\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}} = \boldsymbol{f}^l_{\frac{1}{2},-\frac{1}{2}}(r)\mathfrak{M}^l_{-\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0),\\ \dot{\psi}_1 &= \dot{\psi}_{\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}} = \boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)\mathfrak{M}^{\dot{l}}_{\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),\\ \dot{\psi}_2 &= \dot{\psi}_{\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}} = \boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*)\mathfrak{M}^{\dot{l}}_{-\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),\end{aligned}$$

Substituting these functions into (131) and separating the variables with the aid of recurrence relations between hyperspherical functions, we come to the following system of ordinary

differential equations (the system (121) at $l=1/2$):

$$
\begin{aligned}
-\frac{1}{2}\frac{d\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)}{dr^*}+\frac{1}{4r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)+\frac{\dot{l}+\frac{1}{2}}{2r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*)-\kappa^c\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)=0,\\
\frac{1}{2}\frac{d\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*)}{dr^*}-\frac{1}{4r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*)-\frac{\dot{l}+\frac{1}{2}}{2r^*}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)-\kappa^c\boldsymbol{f}^{l}_{\frac{1}{2},-\frac{1}{2}}(r)=0,\\
\frac{1}{2}\frac{d\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)}{dr}-\frac{1}{4r}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)-\frac{l+\frac{1}{2}}{2r}\boldsymbol{f}^{l}_{\frac{1}{2},-\frac{1}{2}}-\dot{\kappa}^c\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)=0,\\
-\frac{1}{2}\frac{d\boldsymbol{f}^{l}_{\frac{1}{2},-\frac{1}{2}}(r)}{dr}+\frac{1}{4r}\boldsymbol{f}^{l}_{\frac{1}{2},-\frac{1}{2}}(r)+\frac{l+\frac{1}{2}}{2r}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}-\dot{\kappa}^c\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*)=0,
\end{aligned}
$$

For the brevity of exposition we suppose $\boldsymbol{f}_1=\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(r)$, $\boldsymbol{f}_2=\boldsymbol{f}^{l}_{\frac{1}{2},-\frac{1}{2}}(r)$, $\boldsymbol{f}_3=\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)$, $\boldsymbol{f}_4=\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*)$. Then

$$
\begin{aligned}
-2\frac{d\boldsymbol{f}_3}{dr^*}+\frac{1}{r^*}\boldsymbol{f}_3+\frac{2\left(\dot{l}+\frac{1}{2}\right)}{r^*}\boldsymbol{f}_4-4\kappa^c\boldsymbol{f}_1=0,\\
2\frac{d\boldsymbol{f}_4}{dr^*}-\frac{1}{r^*}\boldsymbol{f}_4-\frac{2\left(\dot{l}+\frac{1}{2}\right)}{r^*}\boldsymbol{f}_3-4\kappa^c\boldsymbol{f}_2=0,\\
2\frac{d\boldsymbol{f}_1}{dr}-\frac{1}{r}\boldsymbol{f}_1-\frac{2\left(l+\frac{1}{2}\right)}{r}\boldsymbol{f}_2-4\dot{\kappa}^c\boldsymbol{f}_3=0,\\
-2\frac{d\boldsymbol{f}_2}{dr}+\frac{1}{r}\boldsymbol{f}_2+\frac{2\left(l+\frac{1}{2}\right)}{r}\boldsymbol{f}_1-4\dot{\kappa}^c\boldsymbol{f}_4=0.
\end{aligned}
$$

Let us assume that $\boldsymbol{f}_3=\mp\boldsymbol{f}_4$ and $\boldsymbol{f}_2=\pm\boldsymbol{f}_1$, then the first equation coincides with the second, and the third equations coincides with the fourth. Therefore,

$$
\begin{aligned}
\frac{d\boldsymbol{f}_4}{dr^*}+\frac{\dot{l}}{r^*}\boldsymbol{f}_4-2\kappa^c\boldsymbol{f}_1=0,\\
\frac{d\boldsymbol{f}_1}{dr}-\frac{l+1}{r}\boldsymbol{f}_1+2\dot{\kappa}^c\boldsymbol{f}_4=0.
\end{aligned}
$$

Let us consider a real part $\operatorname{Re} r$ of the radius of complex two-sphere. It is obvious that $\operatorname{Re} r=\operatorname{Re} r^*$. Denoting $\mathsf{z}=\operatorname{Re} r=\operatorname{Re} r^*$ and excluding the function $\boldsymbol{f}_4$ at $l=\dot{l}$, we come to the following differential equation:

$$
\mathsf{z}^2\frac{d^2\boldsymbol{f}_1}{d\mathsf{z}^2}-\mathsf{z}\frac{d\boldsymbol{f}_1}{d\mathsf{z}}-(l^2-1-4\kappa^c\dot{\kappa}^c\mathsf{z}^2)\boldsymbol{f}_1=0. \tag{132}
$$

The latter equation is solvable in the Bessel functions of half-integer order:

$$
\boldsymbol{f}_1(\mathsf{z})=C_1\sqrt{\kappa^c\dot{\kappa}^c}\mathsf{z}J_l\left(\sqrt{\kappa^c\dot{\kappa}^c}\mathsf{z}\right)+C_2\sqrt{\kappa^c\dot{\kappa}^c}\mathsf{z}J_{-l}\left(\sqrt{\kappa^c\dot{\kappa}^c}\mathsf{z}\right).
$$

Further, using recurrence relations between Bessel functions, we find

$$
\begin{aligned}
\boldsymbol{f}_4(\mathsf{z})=\frac{1}{2\kappa^c}\left(\frac{l+1}{\mathsf{z}}\boldsymbol{f}_1(\mathsf{z})-\frac{d\boldsymbol{f}_1}{d\mathsf{z}}\right)=\\
=\frac{C_1}{2}\sqrt{\frac{\dot{\kappa}^c}{\kappa^c}}\mathsf{z}J_{l+1}\left(\sqrt{\kappa^c\dot{\kappa}^c}\mathsf{z}\right)-\frac{C_2}{2}\sqrt{\frac{\dot{\kappa}^c}{\kappa^c}}\mathsf{z}J_{-l-1}\left(\sqrt{\kappa^c\dot{\kappa}^c}\mathsf{z}\right).
\end{aligned}
$$

Therefore,

$$\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r)=C_1\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r J_l\left(\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r\right)+C_2\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r J_{-l}\left(\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r\right),$$

$$\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*)=\frac{C_1}{2}\sqrt{\frac{\dot{\kappa}^c}{\kappa^c}}\operatorname{Re} r^* J_{l+1}\left(\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r^*\right)-\frac{C_2}{2}\sqrt{\frac{\dot{\kappa}^c}{\kappa^c}}\operatorname{Re} r^* J_{-l-1}\left(\sqrt{\kappa^c\dot{\kappa}^c}\operatorname{Re} r^*\right).$$

In such a way, solutions of the system (131) are defined by the following functions:

$$\begin{aligned}
\psi_1(r,\varphi^c,\theta^c) &= \boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r)\mathfrak{M}^{l}_{\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0),\\
\psi_2(r,\varphi^c,\theta^c) &= \pm\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r)\mathfrak{M}^{l}_{-\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\psi}_1(r^*,\dot{\varphi}^c,\dot{\theta}^c) &= \mp\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*)\mathfrak{M}^{\dot{l}}_{\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\psi}_2(r^*,\dot{\varphi}^c,\dot{\theta}^c) &= \boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*)\mathfrak{M}^{\dot{l}}_{-\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),
\end{aligned}$$

where

$$l=\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ldots;\quad n=-l,\ -l+1,\ \ldots,\ l;$$
$$\dot{l}=\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ldots;\quad \dot{n}=-\dot{l},\ -\dot{l}+1,\ \ldots,\ \dot{l},$$

$$\mathfrak{M}^{l}_{\pm\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0)=e^{\mp\frac{1}{2}(\epsilon+i\varphi)}Z^{l}_{\pm\frac{1}{2},n}(\theta,\tau),$$

$$Z^{l}_{\pm\frac{1}{2},n}(\theta,\tau)=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum^{l}_{k=-l}i^{\pm\frac{1}{2}-k}\tan^{\pm\frac{1}{2}-k}\frac{\theta}{2}\tanh^{n-k}\frac{\tau}{2}\times$$
$${}_2F_1\left(\begin{array}{c}\pm\frac{1}{2}-l+1,1-l-k\\ \pm\frac{1}{2}-k+1\end{array}\middle|\, i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{array}{c}n-l+1,1-l-k\\ n-k+1\end{array}\middle|\,\tanh^2\frac{\tau}{2}\right),$$

$$\mathfrak{M}^{\dot{l}}_{\pm\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0)=e^{\mp\frac{1}{2}(\epsilon-i\varphi)}Z^{\dot{l}}_{\pm\frac{1}{2},\dot{n}}(\theta,\tau),$$

$$Z^{\dot{l}}_{\pm\frac{1}{2},\dot{n}}(\theta,\tau)=\cos^{2\dot{l}}\frac{\theta}{2}\cosh^{2\dot{l}}\frac{\tau}{2}\sum^{\dot{l}}_{\dot{k}=-\dot{l}}i^{\pm\frac{1}{2}-\dot{k}}\tan^{\pm\frac{1}{2}-\dot{k}}\frac{\theta}{2}\tanh^{\dot{n}-\dot{k}}\frac{\tau}{2}\times$$
$${}_2F_1\left(\begin{array}{c}\pm\frac{1}{2}-\dot{l}+1,1-\dot{l}-\dot{k}\\ \pm\frac{1}{2}-\dot{k}+1\end{array}\middle|\, i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{array}{c}\dot{n}-\dot{l}+1,1-\dot{l}-\dot{k}\\ \dot{n}-\dot{k}+1\end{array}\middle|\,\tanh^2\frac{\tau}{2}\right).$$

Therefore, in accordance with the factorization (91), an explicit form of the particular solutions of the system (127)–(128) are given by expressions

$$\begin{aligned}
\psi^{l}_{1n}(\boldsymbol{\alpha}) &= \psi^{+}_1(x)\psi^{l}_{1n}(\mathfrak{g})=u_1(\mathbf{p})e^{-ipx}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r)\mathfrak{M}^{l}_{\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0),\\
\psi^{l}_{2n}(\boldsymbol{\alpha}) &= \psi^{+}_2(x)\psi^{l}_{2n}(\mathfrak{g})=\pm u_2(\mathbf{p})e^{-ipx}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r)\mathfrak{M}^{l}_{-\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\psi}^{\dot{l}}_{1\dot{n}}(\boldsymbol{\alpha}) &= \psi^{-}_1(x)\dot{\psi}^{\dot{l}}_{1\dot{n}}(\mathfrak{g})=\mp v_1(\mathbf{p})e^{ipx}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*)\mathfrak{M}^{\dot{l}}_{\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\psi}^{\dot{l}}_{2\dot{n}}(\boldsymbol{\alpha}) &= \psi^{-}_2(x)\dot{\psi}^{\dot{l}}_{2\dot{n}}(\mathfrak{g})=v_2(\mathbf{p})e^{ipx}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*)\mathfrak{M}^{\dot{l}}_{-\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0).
\end{aligned}\qquad(133)$$

The general solutions we obtain via an expansion in the particular solutions:

$$\psi_1(\boldsymbol{\alpha}) = \sum_{p=-\infty}^{+\infty} u_1(\mathbf{p}) e^{ipx} \sum_{l=\frac{1}{2}}^{\infty} \boldsymbol{f}^l_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r) \sum_{n=-l}^{l} \alpha^{\frac{1}{2}}_{l,n} \mathfrak{M}^l_{\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0), \tag{134}$$

$$\psi_2(\boldsymbol{\alpha}) = \pm \sum_{p=-\infty}^{+\infty} u_2(\mathbf{p}) e^{ipx} \sum_{l=\frac{1}{2}}^{\infty} \boldsymbol{f}^l_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} r) \sum_{n=-l}^{l} \alpha^{-\frac{1}{2}}_{l,n} \mathfrak{M}^l_{-\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0), \tag{135}$$

$$\dot{\psi}_1(\boldsymbol{\alpha}) = \mp \sum_{p=-\infty}^{+\infty} v_1(\mathbf{p}) e^{-ipx} \sum_{\dot{l}=\frac{1}{2}}^{\infty} \boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*) \sum_{\dot{n}=-\dot{l}}^{\dot{l}} \alpha^{\frac{1}{2}}_{\dot{l},\dot{n}} \mathfrak{M}^{\dot{l}}_{\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0), \tag{136}$$

$$\dot{\psi}_2(\boldsymbol{\alpha}) = \sum_{p=-\infty}^{+\infty} v_2(\mathbf{p}) e^{-ipx} \sum_{\dot{l}=\frac{1}{2}}^{\infty} \boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} r^*) \sum_{\dot{n}=-\dot{l}}^{\dot{l}} \alpha^{-\frac{1}{2}}_{\dot{l},\dot{n}} \mathfrak{M}^{\dot{l}}_{-\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0), \tag{137}$$

where

$$\alpha^{\pm\frac{1}{2}}_{l,n} = \frac{(-1)^n(2l+1)(2\dot{l}+1)}{32\pi^4 \boldsymbol{f}^l_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re} a)} \int\limits_{\mathbb{S}^2}\int\limits_{T_4} F_{\pm\frac{1}{2}}(\boldsymbol{\alpha}) e^{-ipx} \mathfrak{M}^l_{\pm\frac{1}{2},n}(\varphi,\epsilon,\theta,\tau,0,0) d^4x d^4\mathfrak{g},$$

$$\alpha^{\pm\frac{1}{2}}_{\dot{l},\dot{n}} = \frac{(-1)^{\dot{n}}(2l+1)(2\dot{l}+1)}{32\pi^4 \boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re} a^*)} \int\limits_{\mathbb{S}^2}\int\limits_{T_4} \dot{F}_{\pm\frac{1}{2}}(\boldsymbol{\alpha}) e^{ipx} \mathfrak{M}^{\dot{l}}_{\pm\frac{1}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0) d^4x d^4\mathfrak{g},$$

These series give a solution of the boundary value problem for the Dirac field.

8.1 Quantization

It is well known that the method of second quantization, introduced in the works [23, 24, 34, 35, 54, 55], is a conceptual base of quantum field theory. In connection with the problem of constructing the quantum electrodynamics on the Poincaré group, it needs to transfer the formalism of second quantization onto the group manifold $\mathcal{M}_{10}$ ($\mathcal{M}_8$ or $\mathcal{M}_6$). We consider briefly this question in case of $\mathcal{M}_{10}$.

Let $\mathcal{H}$ be a complex Hilbert space realized with the aid of functions f with summable square on the set M endowed with a measure. Let $f \to f^+$ be an involution on the space $\mathcal{H}$, that is, the mapping satisfying the following conditions:
1) $(f^+)^* = f$,
2) $(f_1 + f_2)^+ = f_1^+ + f_2^+$,
3) $(\lambda f)^+ = \overline{\lambda} f^+$,
4) $(f_1, f_2)^+ = (f_2^+, f_1^+)$.
Let M be a parameter set of the Poincaré group. When a physical system consists of n particles and the states of k-th particle are described by a space $\mathcal{H}_k$ of the functions with summable square on the set M_k, then the states of the system are described by the functions with summable square which depend on n variables $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$, where $\boldsymbol{\alpha}_k \in M_k$. Let us denote this space as $\mathfrak{H}^n$. When the system consists of n identical particles, the sets M_k coincide with each other, and the space $\mathfrak{H}^n$ in this case is called *a configuration space*.

Let $\mid \Psi\rangle$ and $\mid \Phi\rangle$ be state vectors describing the system of n identical particles. The

scalar product of these vectors in the configuration space has a following form

$$(\Psi,\Phi)=\int d\boldsymbol{\alpha}_1\int d\boldsymbol{\alpha}_2\cdots\int d\boldsymbol{\alpha}_n\langle\Psi\mid\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n\rangle\langle\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n\mid\Phi\rangle=$$
$$=\int d\boldsymbol{\alpha}_1\int d\boldsymbol{\alpha}_2\cdots\int d\boldsymbol{\alpha}_n\overline{\Psi}(\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n)\Phi(\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n),\quad(138)$$

where $d\boldsymbol{\alpha}_i$ is a Haar measure on the Poincaré group of the form (92).

It is well known that in dependence on the kind of particles the system of n identical particles is described either by a subspace $\mathcal{H}^n_F\subset\mathcal{H}^n$ of antisymmetric functions or by a subspace $\mathcal{H}^n_B\subset\mathcal{H}^n$ of symmetric functions. In the first case the particles are called *fermions*, in the second case we have *bosons*.

The states of the system, consisting of a variable number of particles, are described by vectors of a space $\mathfrak{H}$. The space $\mathfrak{H}$ is a direct sum of all $\mathfrak{H}^n$ and a one-dimensional space $\mathfrak{H}^0$ which corresponds to the absence of particles (vacuum). The system, consisting of a variable number of fermions, is described by a subspace $\mathcal{H}_F\subset\mathfrak{H}$, where $\mathcal{H}_F=\sum\limits^{\infty}_{n=0}\oplus\mathcal{H}^n_F$, $\mathcal{H}^0_F=\mathfrak{H}^0$. In turn, the system, consisting of a variable number of bosons, is described by a subspace $\mathcal{H}_B\subset\mathfrak{H}$, where $\mathcal{H}_B=\sum\limits^{\infty}_{n=0}\mathcal{H}^n_B$, $\mathcal{H}^0_B=\mathfrak{H}^0$. The elements of the subspaces $\mathcal{H}_F$ and $\mathcal{H}_B$ describe the states of real physical systems. By this reason, $\mathcal{H}_F$ and $\mathcal{H}_B$ are called *state spaces* (or *Fock spaces*).

The vectors $\mid\Psi\rangle$ of $\mathfrak{H}$ can be written in the following form:

$$\mid\Psi\rangle=\begin{bmatrix}\langle 0\mid\Psi\rangle\\ \langle\boldsymbol{\alpha}_1\mid\Psi\rangle\\ \langle\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2\mid\Psi\rangle\\ \vdots\\ \langle\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\ldots,\boldsymbol{\alpha}_n;n\mid\Psi\rangle\\ \vdots\end{bmatrix}=\begin{bmatrix}\Psi^0\\ \Psi^1(\boldsymbol{\alpha}_1)\\ \Psi^2(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)\\ \vdots\\ \Psi^n(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\ldots,\boldsymbol{\alpha}_n)\\ \vdots\end{bmatrix}\quad(139)$$

and

$$(\Psi,\Psi)=\left|\Psi^0\right|^2+\sum^{\infty}_{n=1}\int\left|\Psi^n(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\ldots,\boldsymbol{\alpha}_n)\right|^2d^n\boldsymbol{\alpha},\quad(140)$$

where $d^n\boldsymbol{\alpha}=d\boldsymbol{\alpha}_1d\boldsymbol{\alpha}_2\cdots d\boldsymbol{\alpha}_n$. It is obvious that the vectors, which have only n-th nonzero components, form the subspace $\mathfrak{H}^n\subset\mathfrak{H}$. In turn, the vectors, belonging to the spaces $\mathcal{H}_F$ and $\mathcal{H}_B$, and also their scalar products in these spaces, are defined by the formulae (139) and (140), but in the first case all the functions $\Psi^n(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\ldots,\boldsymbol{\alpha}_n)$ are antisymmetric, and in the second case they are symmetric.

The action of annihilation and creation operators on the n-particle states are defined by the standard formulae

$$\begin{aligned}\mathsf{a}_i\mid n_1,n_2,\ldots,n_i,\ldots\rangle&=\sqrt{n_i}\mid n_1,n_2,\ldots,n_i-1,\ldots\rangle,\\ \mathsf{a}^+_i\mid n_1,n_2,\ldots,n_i,\ldots\rangle&=\sqrt{n_i+1}\mid n_1,n_2,\ldots,n_i+1,\ldots\rangle\end{aligned}$$

in the case of Bose-Einstein statistics, and

$$\begin{aligned}\mathsf{a}_i\mid n_1,n_2,\ldots,n_i,\ldots\rangle&=(-1)^{s_i}n_i\mid n_1,n_2,\ldots,n_i-1,\ldots\rangle,\\ \mathsf{a}^+_i\mid n_1,n_2,\ldots,n_i,\ldots\rangle&=(-1)^{s_i}(1-n_i)\mid n_1,n_2,\ldots,n_i+1,\ldots\rangle\end{aligned}$$

in the case of Fermi-Dirac statistics.

In like manner, the formalism of second quantization can be transferred onto the homogeneous spaces $\mathcal{M}_8$, $\mathcal{M}_7$ and $\mathcal{M}_6$, and also onto other homogeneous spaces of the Poincaré group contained in the Finkelstein-Bacry-Kihlberg list [32, 5] and endowed with a measure.

Coming back to the electron-positron field, we see that the following logical step consists in definition of the field operators. As is known, in the standard quantum field theory the field operators are defined in the form of Fourier integrals (or Fourier series) on functions, which, in general, are solutions of some relativistic wave equation in the plane wave approximation.

In our case the electron-positron field can be represented in the form of following superpositions (field operators):

$$\begin{aligned}
\boldsymbol{\psi}(\boldsymbol{\alpha}) &= \sum_{s=1}^{2}\mathsf{a}_s\psi_s(\boldsymbol{\alpha})+\sum_{s=1}^{2}\mathsf{b}_s^+\dot{\psi}_s(\boldsymbol{\alpha}),\\
\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}) &= \sum_{s=1}^{2}\mathsf{a}_s^+\overline{\psi}_s(\boldsymbol{\alpha})+\sum_{s=1}^{2}\mathsf{b}_s\overline{\dot{\psi}}_s(\boldsymbol{\alpha}),
\end{aligned}\tag{141}$$

where a_s^+ and a_s are creation and annihilation operators of the electron in a state s, b_s^+ and b_s are creation and annihilation operators of the positron in a state s, $\psi_s(\boldsymbol{\alpha})$ and $\dot{\psi}_s(\boldsymbol{\alpha})$ are Fourier series (134)–(135) and (136)–(137) which form a bispinor $(\psi_1(\boldsymbol{\alpha}),\psi_2(\boldsymbol{\alpha}),\dot{\psi}_1(\boldsymbol{\alpha}),\dot{\psi}_2(\boldsymbol{\alpha}))^T$ on the homogeneous space $\mathcal{M}_8$. The operators a_s^+, a_s, b_s^+, b_s satisfy the following anticommutation relations:

$$\begin{aligned}
\left[\mathsf{a}_s,\mathsf{a}_{s'}^+\right]_+ &= \delta_{ss'}, & \left[\mathsf{b}_s,\mathsf{b}_{s'}^+\right]_+ &= \delta_{ss'},\\
\left[\mathsf{a}_s,\mathsf{a}_{s'}\right]_+ &= 0, & \left[\mathsf{b}_s,\mathsf{b}_{s'}\right]_+ &= 0,\\
\left[\mathsf{a}_s^+,\mathsf{a}_{s'}^+\right]_+ &= 0, & \left[\mathsf{b}_s^+,\mathsf{b}_{s'}^+\right]_+ &= 0,
\end{aligned}\tag{142}$$
$$\left[\mathsf{a}_s,\mathsf{b}_{s'}\right]_+=\left[\mathsf{a}_s,\mathsf{b}_{s'}^+\right]_+=\left[\mathsf{a}_s^+,\mathsf{b}_{s'}\right]_+=\left[\mathsf{a}_s^+,\mathsf{b}_{s'}^+\right]_+=0.$$

Using the relations (142), we can calculate anticommutators of the electron-positron field:

$$\begin{aligned}
\left[\boldsymbol{\psi}_\alpha(\boldsymbol{\alpha}),\boldsymbol{\psi}_\beta(\boldsymbol{\alpha}')\right]_+ &= 0,\\
\left[\overline{\boldsymbol{\psi}}_\alpha(\boldsymbol{\alpha}),\overline{\boldsymbol{\psi}}_\beta(\boldsymbol{\alpha}')\right]_+ &= 0,\\
\left[\boldsymbol{\psi}_\alpha(\boldsymbol{\alpha}),\overline{\boldsymbol{\psi}}_\beta(\boldsymbol{\alpha}')\right]_+ &= S_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}'),
\end{aligned}\tag{143}$$

where

$$\begin{aligned}
S_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}') &= S^+_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}')+S^-_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}'),\\
S^+_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}') &= \sum_{s=1}^{2}\psi_{s\alpha}(\boldsymbol{\alpha})\overline{\psi}_{s\beta}(\boldsymbol{\alpha}'),\\
S^-_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}') &= \sum_{s=1}^{2}\dot{\psi}_{s\alpha}(\boldsymbol{\alpha})\overline{\dot{\psi}}_{s\beta}(\boldsymbol{\alpha}').
\end{aligned}$$

In the absence of an external electromagnetic field the functions $S^+_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}')$ and $S^-_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}')$

can be written as follows

$$S^{+}_{\alpha\beta}(\boldsymbol{\alpha}-\boldsymbol{\alpha}')=\sum^{2}_{s=1}\sum_{p}u_{\alpha s}(\mathbf{p})\overline{u}_{\beta s}(\mathbf{p})e^{ip(x-x')}\times$$
$$\sum^{\infty}_{l=\frac{1}{2}}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re}r)\overline{\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re}r)}\times$$
$$\sum^{l}_{n=-l}\mathfrak{M}^{l}_{\frac{(-1)^{s-1}}{2},n}(\varphi,\epsilon,\theta,\tau,0,0)\overline{\mathfrak{M}^{l}_{\frac{(-1)^{s-1}}{2},n}(\varphi',\epsilon',\theta',\tau',0,0)},$$

$$S^{-}_{\alpha\beta}(\boldsymbol{\alpha}-\boldsymbol{\alpha}')=\sum^{2}_{s=1}\sum_{p}v_{\alpha s}(\mathbf{p})\overline{v}_{\beta s}(\mathbf{p})e^{-ip(x-x')}\times$$
$$\sum^{\infty}_{\dot{l}=\frac{1}{2}}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re}r^{*})\overline{\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re}r^{*})}\times$$
$$\sum^{\dot{l}}_{\dot{n}=-\dot{l}}\mathfrak{M}^{\dot{l}}_{\frac{(-1)^{s-1}}{2},\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0)\overline{\mathfrak{M}^{\dot{l}}_{\frac{(-1)^{s-1}}{2},\dot{n}}(\varphi',\epsilon',\theta',\tau',0,0)},$$

Taking into account that $\mathfrak{M}^{l}_{mn}(\varphi,\epsilon,\theta,\tau,0,0)=e^{-m\varphi^{c}}Z^{l}_{mn}(\cos\theta^{c})$, we obtain $\overline{\mathfrak{M}^{l}_{mn}(\varphi,\epsilon,\theta,\tau,0,0)}=e^{im\varphi^{c}}\overline{Z^{l}_{mn}(\cos\theta^{c})}=(-1)^{n-m}e^{im\varphi^{c}}Z^{l}_{nm}(\cos\theta^{c})$. Using the addition theorem for hyperspherical functions (see [117]), we obtain

$$S^{+}_{\alpha\beta}(\boldsymbol{\alpha}-\boldsymbol{\alpha}')=\sum^{2}_{s=1}\sum_{p}u_{\alpha s}(\mathbf{p})\overline{u}_{\beta s}(\mathbf{p})e^{ip(x-x')}\times$$
$$e^{-\frac{i(-1)^{s-1}}{2}(\varphi^{-}\varphi^{c\prime})}\sum^{\infty}_{l=\frac{1}{2}}\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re}r)\overline{\boldsymbol{f}^{l}_{\frac{1}{2},\frac{1}{2}}(\operatorname{Re}r)}Z^{l}_{\frac{(-1)^{s-1}}{2},\frac{(-1)^{s-1}}{2}}(\cos\theta^{c\prime\prime}),$$

$$S^{-}_{\alpha\beta}(\boldsymbol{\alpha}-\boldsymbol{\alpha}')=\sum^{2}_{s=1}\sum_{p}v_{\alpha s}(\mathbf{p})\overline{v}_{\beta s}(\mathbf{p})e^{-ip(x-x')}\times$$
$$e^{\frac{i(-1)^{s-1}}{2}(\varphi^{c}-\varphi^{c\prime})}\sum^{\infty}_{\dot{l}=\frac{1}{2}}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re}r^{*})\overline{\boldsymbol{f}^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\operatorname{Re}r^{*})}Z^{\dot{l}}_{\frac{(-1)^{s-1}}{2},\frac{(-1)^{s-1}}{2}}(\cos\dot{\theta}^{c\prime\prime}),$$

where $\cos\theta^{c\prime\prime}=\cos\theta^{c}\cos\theta^{c\prime}-\sin\theta^{c}\sin\theta^{c\prime}\cos\varphi^{c\prime}$.

Let us consider now normal products of the operators (141). Supposing

$$\boldsymbol{\psi}(\boldsymbol{\alpha})=\psi^{(+)}(\boldsymbol{\alpha})+\psi^{(-)}(\boldsymbol{\alpha}),\quad\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})=\overline{\psi}^{(+)}(\boldsymbol{\alpha})+\overline{\psi}^{(-)}(\boldsymbol{\alpha}),$$

where $\psi^{(+)}(\boldsymbol{\alpha})$ are annihilation operators of electrons and $\psi^{(-)}(\boldsymbol{\alpha})$ are creation operators of positrons, and correspondingly $\overline{\psi}^{(+)}(\boldsymbol{\alpha})$ are creation operators of electrons and $\overline{\psi}^{(-)}(\boldsymbol{\alpha})$ are

annihilation operators of positrons, we see that

$$
\begin{aligned}
N(\psi^{(+)}(\boldsymbol{\alpha})\overline{\psi}^{(+)}(\boldsymbol{\alpha}')) &= -\overline{\psi}^{(+)}(\boldsymbol{\alpha}')\psi^{(+)}(\boldsymbol{\alpha}),\\
N(\psi^{(-)}(\boldsymbol{\alpha})\overline{\psi}^{(-)}(\boldsymbol{\alpha}')) &= \psi^{(-)}(\boldsymbol{\alpha})\overline{\psi}^{(-)}(\boldsymbol{\alpha}'),\\
N(\psi^{(+)}(\boldsymbol{\alpha})\overline{\psi}^{(-)}(\boldsymbol{\alpha}')) &= \psi^{(+)}(\boldsymbol{\alpha})\overline{\psi}^{(-)}(\boldsymbol{\alpha}'),\\
N(\psi^{(-)}(\boldsymbol{\alpha})\overline{\psi}^{(+)}(\boldsymbol{\alpha}')) &= \psi^{(-)}(\boldsymbol{\alpha})\overline{\psi}^{(+)}(\boldsymbol{\alpha}').
\end{aligned}
$$

Taking into account the latter N-products, we have

$$
\begin{aligned}
\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}') = (\psi^{(+)}(\boldsymbol{\alpha}) + \psi^{(-)}(\boldsymbol{\alpha}))(\overline{\psi}^{(+)}(\boldsymbol{\alpha}') + \overline{\psi}^{(-)}(\boldsymbol{\alpha}')) = \\
= \psi^{(+)}(\boldsymbol{\alpha})\overline{\psi}^{(+)}(\boldsymbol{\alpha}') + \psi^{(-)}(\boldsymbol{\alpha})\overline{\psi}^{(+)}(\boldsymbol{\alpha}') + \psi^{(+)}(\boldsymbol{\alpha})\overline{\psi}^{(-)}(\boldsymbol{\alpha}') + \\
+ \psi^{(-)}(\boldsymbol{\alpha})\overline{\psi}^{(-)}(\boldsymbol{\alpha}').
\end{aligned}
$$

In accordance with (143), $\psi^{(+)}(\boldsymbol{\alpha})\overline{\psi}^{(+)}(\boldsymbol{\alpha}') = -\overline{\psi}^{(+)}(\boldsymbol{\alpha}')\psi^{(+)}(\boldsymbol{\alpha}) + S^{+}(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$, therefore,

$$
\begin{aligned}
\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}') &= N(\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')) + S^{+}(\boldsymbol{\alpha} - \boldsymbol{\alpha}')\\
&= N(\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')) + \underbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}'),
\end{aligned}
\tag{144}
$$

where $\underbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}') = S^{+}(\boldsymbol{\alpha} - \boldsymbol{\alpha}')$ is an operator coupling. Analogously,

$$
\begin{aligned}
\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}') &= N(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}')) + S^{-}(\boldsymbol{\alpha} - \boldsymbol{\alpha}')\\
&= N(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}')) + \underbracket{\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}}(\boldsymbol{\alpha}'),
\end{aligned}
$$

$$
\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}') = N(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')), \quad \boldsymbol{\psi}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}') = N(\boldsymbol{\psi}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}')).
$$

Hence it follows that couplings of distinct N-products of the operators $\boldsymbol{\psi}$ and $\overline{\boldsymbol{\psi}}$ are

$$
\begin{aligned}
\underbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}') &= S^{+}(\boldsymbol{\alpha} - \boldsymbol{\alpha}');\\
\underbracket{\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}}(\boldsymbol{\alpha}') &= S^{-}(\boldsymbol{\alpha} - \boldsymbol{\alpha}');
\end{aligned}
$$

$$
\underbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\boldsymbol{\psi}}(\boldsymbol{\alpha}') = 0; \quad \underbracket{\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}') = 0.
$$

Thus, the vacuum expectation values of the operator products are defined as

$$
\begin{aligned}
\langle 0 \mid \boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}') \mid 0\rangle &= \underbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}') = S^{+}(\boldsymbol{\alpha} - \boldsymbol{\alpha}'),\\
\langle 0 \mid \overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}') \mid 0\rangle &= \underbracket{\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}}(\boldsymbol{\alpha}') = S^{-}(\boldsymbol{\alpha} - \boldsymbol{\alpha}'),\\
\langle 0 \mid \boldsymbol{\psi}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}') \mid 0\rangle &= 0,\\
\langle 0 \mid \overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}') \mid 0\rangle &= 0.
\end{aligned}
$$

Further, we can define time ordered products of the field operators (since t is a parameter of the Poincaré group):

$$
T(\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')) = \begin{cases} \boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}'), & t > t';\\ -\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')\boldsymbol{\psi}(\boldsymbol{\alpha}), & t' > t. \end{cases} \tag{145}
$$

Using (144), we can express (145) via the N-products:

$$T(\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')) = \begin{cases} N(\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')) + \underbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}'), & t > t'; \\ -N(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')\boldsymbol{\psi}(\boldsymbol{\alpha})) - \underbracket{\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}}(\boldsymbol{\alpha}'), & t' > t. \end{cases} \tag{146}$$

Since $-N(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')\boldsymbol{\psi}(\boldsymbol{\alpha})) = N(\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}'))$, then (146) can be rewritten as

$$T(\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')) = N(\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')) + \overbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}'),$$

where $\overbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}')$ is a time ordered coupling of the field operators:

$$\overbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}') = \begin{cases} \underbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}') = S^{+}(\boldsymbol{\alpha} - \boldsymbol{\alpha}'), & t > t'; \\ -\underbracket{\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')\boldsymbol{\psi}}(\boldsymbol{\alpha}) = -S^{-}(\boldsymbol{\alpha} - \boldsymbol{\alpha}'), & t' > t. \end{cases}$$

In like manner we find for other T-products the following expressions

$$\begin{aligned} T(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}')) &= -T(\boldsymbol{\psi}(\boldsymbol{\alpha}')\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})) \\ &= -N(\boldsymbol{\psi}(\boldsymbol{\alpha}')\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})) - \overbracket{\boldsymbol{\psi}(\boldsymbol{\alpha}')\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}), \end{aligned}$$

$$T(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')) = N(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')), \quad T(\boldsymbol{\psi}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}')) = N(\boldsymbol{\psi}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}')),$$

whence it follows that

$$\overbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}') = -\overbracket{\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')\boldsymbol{\psi}}(\boldsymbol{\alpha}), \quad \overbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\boldsymbol{\psi}}(\boldsymbol{\alpha}') = 0, \quad \overbracket{\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}') = 0.$$

Therefore, vacuum expectation values of the time ordered products are defined by expressions

$$\begin{aligned} \langle 0 \mid T(\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')) \mid 0\rangle &= \overbracket{\boldsymbol{\psi}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}'), \\ \langle 0 \mid T(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}')) \mid 0\rangle &= -\overbracket{\boldsymbol{\psi}(\boldsymbol{\alpha}')\overline{\boldsymbol{\psi}}}(\boldsymbol{\alpha}), \\ \langle 0 \mid T(\boldsymbol{\psi}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}')) \mid 0\rangle &= 0, \\ \langle 0 \mid T(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha}')) \mid 0\rangle &= 0. \end{aligned}$$

9 The Maxwell field

In this section we will set up a boundary value problem for the Maxwell field $(1,0)\oplus(0,1)$ (photon field) defined on the homogeneous space $\mathcal{M}_8$. At this point, electromagnetic field should be defined in the Riemann-Silberstein representation [124, 104, 12]. The Riemann-Silberstein (Majorana-Oppenheimer) representation considered during long time by many authors [71, 81, 45, 77, 94, 75, 22, 39]. The interest to this formulation of electrodynamics has grown in recent years [52, 105, 38, 31]. One of the main advantages of this approach lies in the fact that Dirac and Maxwell fields are derived similarly from the Dirac-like Lagrangians[4]. These fields have the analogous mathematical structure, namely, they are the functions on

[4]Moreover, in contrast to the Gupta-Bleuler method, where the non-observable four-potential A_μ is quantized, the Majorana-Oppenheimer electrodynamics deals directly with observable quantities, such as the electric and magnetic fields. It allows one to avoid non-physical degrees of freedom, indefinite metrics and other difficulties connected with the Gupta-Bleuler method.

the Poincaré group. This circumstance allows us to consider the fields $(1/2,0)\oplus(0,1/2)$ and $(1,0)\oplus(0,1)$ on an equal footing, from the one group theoretical viewpoint[5].

We start with the Lagrangian (98) on the group manifold $\mathcal{M}_{10}$. Let us rewrite (98) in the form

$$\boldsymbol{\mathcal{L}}(\boldsymbol{\alpha})=-\frac{1}{2}\left(\overline{\boldsymbol{\phi}}(\boldsymbol{\alpha})\Gamma_\mu\frac{\partial\boldsymbol{\phi}(\boldsymbol{\alpha})}{\partial x_\mu}-\frac{\partial\overline{\boldsymbol{\phi}}(\boldsymbol{\alpha})}{\partial x_\mu}\Gamma_\mu\boldsymbol{\phi}(\boldsymbol{\alpha})\right)-$$
$$-\frac{1}{2}\left(\overline{\boldsymbol{\phi}}(\boldsymbol{\alpha})\Upsilon_\nu\frac{\partial\boldsymbol{\phi}(\boldsymbol{\alpha})}{\partial\mathfrak{g}_\nu}-\frac{\partial\overline{\boldsymbol{\phi}}(\boldsymbol{\alpha})}{\partial\mathfrak{g}_\nu}\Upsilon_\nu\boldsymbol{\phi}(\boldsymbol{\alpha})\right),\quad(147)$$

where $\boldsymbol{\phi}(\boldsymbol{\alpha})=\phi(x)\phi(\mathfrak{g})$ ($\mu=0,1,2,3,\ \nu=1,\ldots,6$), and

$$\Gamma_0=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad\Gamma_1=\begin{pmatrix}0 & -\alpha_1\\ \alpha_1 & 0\end{pmatrix},\quad\Gamma_2=\begin{pmatrix}0 & -\alpha_2\\ \alpha_2 & 0\end{pmatrix},\quad\Gamma_3=\begin{pmatrix}0 & -\alpha_3\\ \alpha_3 & 0\end{pmatrix},\qquad(148)$$

$$\Upsilon_1=\begin{pmatrix}0 & \Lambda^\ast_1\\ \Lambda_1 & 0\end{pmatrix},\quad\Upsilon_2=\begin{pmatrix}0 & \Lambda^\ast_2\\ \Lambda_2 & 0\end{pmatrix},\quad\Upsilon_3=\begin{pmatrix}0 & \Lambda^\ast_3\\ \Lambda_3 & 0\end{pmatrix},\qquad(149)$$

$$\Upsilon_4=\begin{pmatrix}0 & i\Lambda^\ast_1\\ i\Lambda_1 & 0\end{pmatrix},\quad\Upsilon_5=\begin{pmatrix}0 & i\Lambda^\ast_2\\ i\Lambda_2 & 0\end{pmatrix},\quad\Upsilon_6=\begin{pmatrix}0 & i\Lambda^\ast_3\\ i\Lambda_3 & 0\end{pmatrix},\qquad(150)$$

where

$$\alpha_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & i\\ 0 & -i & 0\end{pmatrix},\quad\alpha_2=\begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix},\quad\alpha_3=\begin{pmatrix}0 & i & 0\\ -i & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad(151)$$

and the matrices Λ_j and $\Lambda^\ast_j$ are derived from (111)–(110) and (113)–(115) at $l=1$:

$$\Lambda_1=\frac{c_{11}}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\quad\Lambda_2=\frac{c_{11}}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix},\quad\Lambda_3=c_{11}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}.\qquad(152)$$

$$\Lambda^\ast_1=\frac{\sqrt{2}}{2}\dot{c}_{11}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},\quad\Lambda^\ast_2=\frac{\sqrt{2}}{2}\dot{c}_{11}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix},\quad\Lambda^\ast_3=\dot{c}_{11}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}.\qquad(153)$$

Varying independently $\phi(x)$ and $\overline{\phi}(x)$ in the Lagrangian (147), and then $\phi(\mathfrak{g})$ and $\overline{\phi}(\mathfrak{g})$, we come to the following equations:

$$\Gamma_\mu\frac{\partial\phi(x)}{\partial x_\mu}=0,\qquad(154)$$
$$\Gamma^T_\mu\frac{\partial\overline{\phi}(x)}{\partial x_\mu}=0.\qquad(155)$$
$$\Upsilon_\nu\frac{\partial\phi(\mathfrak{g})}{\partial\mathfrak{g}_\nu}=0,\qquad(156)$$
$$\Upsilon^T_\nu\frac{\partial\overline{\phi}(\mathfrak{g})}{\partial\mathfrak{g}_\nu}=0.\qquad(157)$$

[5]In this connection it is interesting to note that in the gauge theories electromagnetic field is understood as a ‘gauge field’ that leads to a peculiar opposition with other physical fields called by this reason as ‘matter fields’.

Let us formulate the boundary value problem for the field $(1,0)\oplus(0,1)$. *Let T be an unbounded region in $\mathcal{M}_8=\mathbb{R}^{1,3}\times\mathbb{S}^2$ and let Σ ($\dot{\Sigma}$) be a surface of the complex (dual) two-sphere, then it needs to find the functions $\boldsymbol{\phi}(\boldsymbol{\alpha})=(\phi_1(\boldsymbol{\alpha}),\phi_2(\boldsymbol{\alpha}),\phi_3(\boldsymbol{\alpha}),\dot{\phi}_1(\boldsymbol{\alpha}),\dot{\phi}_2(\boldsymbol{\alpha}),\dot{\phi}_3(\boldsymbol{\alpha}))^T$, such that*
1) $\boldsymbol{\phi}(\boldsymbol{\alpha})$ satisfies the equations (154)–(155) and (156)–(157) in the all region T.
2) $\boldsymbol{\phi}(\boldsymbol{\alpha})$ is a continuous function everywhere in T.
3) $\phi_m(\boldsymbol{\alpha})\big|_{\Sigma}=F_m(\boldsymbol{\alpha})$, $\dot{\phi}_m(\boldsymbol{\alpha})\Big|_{\dot{\Sigma}}=\dot{F}_m(\boldsymbol{\alpha})$, where $F_m(\boldsymbol{\alpha})$ and $\dot{F}_m(\boldsymbol{\alpha})$ are square integrable and infinitely differentiable functions on $\mathcal{M}_8$, $m=1,2,3$.

First of all, the equation (154) can be written as follows:

$$\left[\frac{i\hbar}{c}\frac{\partial}{\partial t}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}-i\hbar\frac{\partial}{\partial\mathbf{x}}\begin{pmatrix}0 & -\alpha_i\\ \alpha_i & 0\end{pmatrix}\right]\begin{pmatrix}\phi(x)\\ \dot{\phi}(x)\end{pmatrix}=0, \tag{158}$$

where

$$\begin{pmatrix}\phi(x)\\ \dot{\phi}(x)\end{pmatrix}=\begin{pmatrix}\mathbf{E}-i\mathbf{B}\\ \mathbf{E}+i\mathbf{B}\end{pmatrix}=\begin{pmatrix}E_1-iB_1\\ E_2-iB_2\\ E_3-iE_3\\ E_1+iB_1\\ E_2+iB_2\\ E_3+iB_3\end{pmatrix}.$$

From the equation (158) it follows that

$$\left(\frac{i\hbar}{c}\frac{\partial}{\partial t}-i\hbar\alpha_i\frac{\partial}{\partial\mathbf{x}}\right)\phi(x)=0, \tag{159}$$
$$\left(\frac{i\hbar}{c}\frac{\partial}{\partial t}+i\hbar\alpha_i\frac{\partial}{\partial\mathbf{x}}\right)\dot{\phi}(x)=0. \tag{160}$$

The latter equations with allowance for transversality conditions ($\mathbf{p}\cdot\phi=0$, $\mathbf{p}\cdot\dot{\phi}=0$) coincide with the Maxwell equations. Indeed, taking into account that $(\mathbf{p}\cdot\alpha)\phi=\hbar\nabla\times\phi$, we obtain

$$\frac{i\hbar}{c}\frac{\partial\phi}{\partial t} = -\hbar\nabla\times\phi, \tag{161}$$
$$-i\hbar\nabla\cdot\phi = 0. \tag{162}$$

Whence

$$\begin{aligned}\nabla\times(\mathbf{E}-i\mathbf{B}) &= -\frac{i}{c}\frac{\partial(\mathbf{E}-i\mathbf{B})}{\partial t},\\ \nabla\cdot(\mathbf{E}-i\mathbf{B}) &= 0\end{aligned}$$

(the constant $\hbar$ is cancelled). Separating the real and imaginary parts, we obtain Maxwell equations

$$\begin{aligned}\nabla\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\\ \nabla\times\mathbf{B} &= \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t},\\ \nabla\cdot\mathbf{E} &= 0,\\ \nabla\cdot\mathbf{B} &= 0.\end{aligned}$$

It is easy to verify that we come again to Maxwell equations starting from the equations

$$\begin{aligned}
\left(\frac{i\hbar}{c}\frac{\partial}{\partial t} + i\hbar\alpha_i\frac{\partial}{\partial \mathbf{x}}\right)\dot{\phi}(x) &= 0, &(163)\\
-i\hbar\nabla\cdot\dot{\phi}(x) &= 0. &(164)
\end{aligned}$$

In spite of the fact that equations (159) and (160) lead to the same Maxwell equations, the physical interpretation of these equations is different (see [12, 38]). Namely, the equations (159) and (160) are equations with negative and positive helicity, respectively.

As usual, the conjugated wavefunction $\overline{\phi}(x) = \overset{+}{\phi}(x)\Gamma_0 = (\phi(x), \dot{\phi}(x))$ corresponds to antiparticle (it is a direct consequence of the Dirac-like Lagrangian (147), $\phi(x)$ is a complex function). Therefore, we come here to a very controversial conclusion that the equations (155) describe the antiparticle (antiphoton) and, moreover, hence it follows that there exist the current and charge for the photon field. At first glance, we come to a drastic contradiction with the widely accepted fact that the photon is truly neutral particle. However, it is easy to verify that equations (155) lead to the Maxwell equations also. It means that the photon coincides with its "antiparticle". Following to the standard procedure given in many textbooks, we can define the "charge" of the photon by an expression

$$Q \sim \int d\mathbf{x}\overline{\phi}\Gamma_0\phi, \tag{165}$$

where $\overline{\phi}\Gamma_0\phi = 2(\mathbf{E}^2 + \mathbf{B}^2)$. However, Newton and Wigner [79] showed that for the photon there exist no localized states. Therefore, the integral in the right side of (165) presents an indeterminable expression. Since the integral (165) does not exist in general, then the "charge" of the photon cannot be considered as a constant magnitude (as it takes place for the electron field which has localized states [79] and a well-defined constant charge). In a sense, one can say that the "charge " of the photon is equal to the energy $\mathbf{E}^2 + \mathbf{B}^2$ of the γ-quantum.

We see that the equation (158) leads to the two Dirac-like equations (159) and (160) which in combination with the transversality conditions (162) and (164) are equivalent to the Maxwell equations. Let us represent solutions of (159) in a plane-wave form

$$\phi(x) = \varepsilon(\mathbf{k})\exp[i\hbar^{-1}(\mathbf{k}\cdot\mathbf{x} - \omega t)]. \tag{166}$$

After substitution of (166) into (159) we come to the following matrix eigenvalue problem

$$-c\begin{pmatrix} 0 & ik_3 & -ik_2 \\ -ik_3 & 0 & ik_1 \\ ik_2 & -ik_1 & 0 \end{pmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{pmatrix} = \omega\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{pmatrix}.$$

It is easy to verify that we come to the same eigenvalue problem starting from (160). The secular equation has the solutions $\omega = \pm c\mathbf{k}, 0$.

Therefore, solutions of (158) in the plane-wave approximation are expressed via the functions

$$\begin{aligned}
\phi_\pm(\mathbf{k};\mathbf{x},t) &= \left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\begin{pmatrix}\varepsilon_\pm(\mathbf{k})\\ \varepsilon_\pm(\mathbf{k})\end{pmatrix}\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)],\\
\phi_0(\mathbf{k};\mathbf{x}) &= \left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\begin{pmatrix}\varepsilon_0(\mathbf{k})\\ \varepsilon_0(\mathbf{k})\end{pmatrix}\exp[i\mathbf{k}\cdot\mathbf{x}]
\end{aligned}$$

and the complex conjugate functions $\dot{\phi}_+(\mathbf{k};\mathbf{x},t)$ and $\dot{\phi}_0(\mathbf{k};\mathbf{x})$ $(\mathbf{E}+i\mathbf{B})$ corresponding to positive helicity, here $\omega = c|\mathbf{k}|$ and $\varepsilon_\lambda(\mathbf{k})$ $(\lambda=\pm,0)$ are the polarization vectors of a photon:

$$\begin{aligned}
\varepsilon_\pm(\mathbf{k}) &= \left\{2|\mathbf{k}|^2(k_1^2+k_2^2)\right\}^{-\frac{1}{2}}\begin{bmatrix}-k_1k_3\pm ik_2|\mathbf{k}|\\ -k_2k_3\mp ik_1|\mathbf{k}|\\ k_1^2+k_2^2\end{bmatrix},\\
\varepsilon_0(\mathbf{k}) &= |\mathbf{k}|^{-1}\begin{bmatrix}k_1\\ k_2\\ k_3\end{bmatrix}.
\end{aligned}$$

Let us find now solutions of the $SL(2,\mathbb{C})$-part equations (156)–(157). Taking into account the structure of Υ_ν given by (124)–(125), we can rewrite the equation (156) as follows

$$\begin{aligned}
\sum_{k=1}^{3}\Lambda_k\frac{\partial\phi}{\partial a_k}-i\sum_{k=1}^{3}\Lambda_k\frac{\partial\phi}{\partial a_k^*} &= 0,\\
\sum_{k=1}^{3}\Lambda_k^*\frac{\partial\dot{\phi}}{\partial\widetilde{a}_k}+i\sum_{k=1}^{3}\Lambda_k^*\frac{\partial\dot{\phi}}{\partial\widetilde{a}_k^*} &= 0.
\end{aligned}$$

Or, using the explicit form of the matrices Λ_j and Λ_j^* given by (152)–(153), we obtain

$$\begin{aligned}
&\frac{\sqrt{2}}{2}\frac{\partial\phi_2}{\partial a_1}-i\frac{\sqrt{2}}{2}\frac{\partial\phi_2}{\partial a_2}+\frac{\partial\phi_1}{\partial a_3}-i\frac{\sqrt{2}}{2}\frac{\partial\phi_2}{\partial a_1^*}-\frac{\sqrt{2}}{2}\frac{\partial\phi_2}{\partial a_2^*}-i\frac{\partial\phi_1}{\partial a_3^*}=0,\\
&\frac{\partial\phi_1}{\partial a_1}+\frac{\partial\phi_3}{\partial a_1}+i\frac{\partial\phi_1}{\partial a_2}-i\frac{\partial\phi_3}{\partial a_2}-i\frac{\partial\phi_1}{\partial a_1^*}-i\frac{\partial\phi_3}{\partial a_1^*}-\frac{\partial\phi_1}{\partial a_2^*}+\frac{\partial\phi_3}{\partial a_2^*}=0,\\
&\frac{\sqrt{2}}{2}\frac{\partial\phi_2}{\partial a_1}+i\frac{\sqrt{2}}{2}\frac{\partial\phi_2}{\partial a_2}-\frac{\partial\phi_3}{\partial a_3}-i\frac{\sqrt{2}}{2}\frac{\partial\phi_2}{\partial a_1^*}+\frac{\sqrt{2}}{2}\frac{\partial\phi_2}{\partial a_2^*}+i\frac{\partial\phi_3}{\partial a_3^*}=0,\\
&\frac{\sqrt{2}}{2}\frac{\partial\dot{\phi}_2}{\partial\widetilde{a}_1}-i\frac{\sqrt{2}}{2}\frac{\partial\dot{\phi}_2}{\partial\widetilde{a}_2}+\frac{\partial\dot{\phi}_1}{\partial\widetilde{a}_3}+i\frac{\sqrt{2}}{2}\frac{\partial\dot{\phi}_2}{\partial\widetilde{a}_1^*}+\frac{\sqrt{2}}{2}\frac{\partial\dot{\phi}_2}{\partial\widetilde{a}_2^*}+i\frac{\partial\dot{\phi}_1}{\partial\widetilde{a}_3^*}=0,\\
&\frac{\partial\dot{\phi}_1}{\partial\widetilde{a}_1}+\frac{\partial\dot{\phi}_3}{\partial\widetilde{a}_1}+i\frac{\partial\dot{\phi}_1}{\partial\widetilde{a}_2}-i\frac{\partial\dot{\phi}_3}{\partial\widetilde{a}_2}+i\frac{\partial\dot{\phi}_1}{\partial\widetilde{a}_1^*}+i\frac{\partial\dot{\phi}_3}{\partial\widetilde{a}_1^*}-\frac{\partial\dot{\phi}_1}{\partial\widetilde{a}_2^*}+\frac{\partial\dot{\phi}_3}{\partial\widetilde{a}_2^*}=0,\\
&\frac{\sqrt{2}}{2}\frac{\partial\dot{\phi}_2}{\partial\widetilde{a}_1}+i\frac{\sqrt{2}}{2}\frac{\partial\dot{\phi}_2}{\partial\widetilde{a}_2}-\frac{\partial\dot{\phi}_3}{\partial\widetilde{a}_3}+i\frac{\sqrt{2}}{2}\frac{\partial\dot{\phi}_2}{\partial\widetilde{a}_1^*}-\frac{\sqrt{2}}{2}\frac{\partial\dot{\phi}_2}{\partial\widetilde{a}_2^*}-i\frac{\partial\dot{\phi}_3}{\partial\widetilde{a}_3^*}=0,
\end{aligned}\tag{167}$$

Coming to the helicity basis, we will find solutions of the equations (167), that is, we will present components of the Majorana–Oppenheimer 'bispinor' $\boldsymbol{\phi}=(\phi_1,\phi_2,\phi_3,\dot{\phi}_1,\dot{\phi}_2,\dot{\phi}_3)^T$ in terms of the functions on the two-dimensional complex sphere (it is obvious that the indices k and $\dot{k}$ can be omitted here):

$$\begin{aligned}
\phi_1 &= \phi_{1,1;1,1}=\boldsymbol{f}^l_{1,1}(r)\mathfrak{M}^l_{1,n}(\varphi,\epsilon,\theta,\tau,0,0),\\
\phi_2 &= \phi_{1,0;1,0}=\boldsymbol{f}^l_{1,0}(r)\mathfrak{M}^l_{0,n}(\varphi,\epsilon,\theta,\tau,0,0),\\
\phi_3 &= \phi_{1,-1;1,-1}=\boldsymbol{f}^l_{1,-1}(r)\mathfrak{M}^l_{-1,n}(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\phi}_1 &= \dot{\phi}_{1,1;1,1}=\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)\mathfrak{M}^{\dot{l}}_{1,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\phi}_2 &= \dot{\phi}_{1,0;1,0}=\boldsymbol{f}^{\dot{l}}_{1,0}(r^*)\mathfrak{M}^{\dot{l}}_{0,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\phi}_3 &= \dot{\phi}_{1,-1;1,-1}=\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)\mathfrak{M}^{\dot{l}}_{-1,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),
\end{aligned}$$

Substituting these functions into (167) and separating the variables with the aid of recurrence relations between hyperspherical functions, we come to the following system of ordinary differential equations (the system (121) at $l = 1$):

$$
\begin{gathered}
2\frac{d\boldsymbol{f}^l_{1,1}(r)}{dr} - \frac{1}{r}\boldsymbol{f}^l_{1,1}(r) - \frac{\sqrt{2l(l+1)}}{r}\boldsymbol{f}^l_{1,0}(r) = 0,\\
-\frac{\sqrt{2l(l+1)}}{r}\boldsymbol{f}^l_{1,-1}(r) + \frac{\sqrt{2l(l+1)}}{r}\boldsymbol{f}^l_{1,1}(r) = 0,\\
-2\frac{d\boldsymbol{f}^l_{1,-1}(r)}{dr} + \frac{1}{r}\boldsymbol{f}^l_{1,-1}(r) + \frac{\sqrt{2l(l+1)}}{r}\boldsymbol{f}^l_{1,0}(r) = 0,\\
2\frac{d\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)}{dr^*} - \frac{1}{r^*}\boldsymbol{f}^{\dot{l}}_{1,1}(r^*) - \frac{\sqrt{2\dot{l}(\dot{l}+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,0}(r^*) = 0,\\
-\frac{i\sqrt{2\dot{l}(\dot{l}+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*) + \frac{\sqrt{2\dot{l}(\dot{l}+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,1}(r^*) = 0,\\
-2\frac{d\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)}{dr^*} + \frac{1}{r^*}\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*) + \frac{\sqrt{2\dot{l}(\dot{l}+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,0}(r^*) = 0,
\end{gathered}
\tag{168}
$$

From the second and fifth equations it follows that $\boldsymbol{f}^l_{1,-1}(r) = \boldsymbol{f}^l_{1,1}(r)$ and $\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*) = \boldsymbol{f}^{\dot{l}}_{1,1}(r^*)$. Taking into account these relations we can rewrite the system (168) as follows

$$
\begin{array}{rcl}
2\frac{d\boldsymbol{f}^l_{1,1}(r)}{dr} - \frac{1}{r}\boldsymbol{f}^l_{1,1}(r) - \frac{\sqrt{2l(l+1)}}{r}\boldsymbol{f}^l_{1,0}(r) & = & 0,\\
-2\frac{d\boldsymbol{f}^l_{1,-1}(r)}{dr} + \frac{1}{r}\boldsymbol{f}^l_{1,-1}(r) + \frac{\sqrt{2l(l+1)}}{r}\boldsymbol{f}^l_{1,0}(r) & = & 0,\\
2\frac{d\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)}{dr^*} - \frac{1}{r^*}\boldsymbol{f}^{\dot{l}}_{1,1}(r^*) - \frac{\sqrt{2\dot{l}(\dot{l}+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,0}(r^*) & = & 0,\\
-2\frac{d\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)}{dr^*} + \frac{1}{r^*}\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*) + \frac{\sqrt{2\dot{l}(\dot{l}+1)}}{r^*}\boldsymbol{f}^{\dot{l}}_{1,0}(r^*) & = & 0,
\end{array}
\tag{169}
$$

It is easy to see that the first equation is equivalent to the second, and third equation is equivalent to the fourth. Thus, we come to the following inhomogeneous differential equations of the first order:

$$
\begin{gathered}
2r\frac{d\boldsymbol{f}^l_{1,1}(r)}{dr} - \boldsymbol{f}^l_{1,1}(r) - \sqrt{2l(l+1)}\boldsymbol{f}^l_{1,0}(r) = 0,\\
2r^*\frac{d\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)}{dr^*} - \boldsymbol{f}^{\dot{l}}_{1,1}(r^*) - \sqrt{2\dot{l}(\dot{l}+1)}\boldsymbol{f}^{\dot{l}}_{1,0}(r^*) = 0,
\end{gathered}
$$

where the functions $\boldsymbol{f}^l_{1,0}(r)$ and $\boldsymbol{f}^{\dot{l}}_{1,0}(r^*)$ are understood as inhomogeneous parts. Solutions of these equations are expressed via the elementary functions:

$$
\begin{array}{rcl}
\boldsymbol{f}^l_{1,1}(r) & = & C\sqrt{r} + \sqrt{2l(l+1)}r,\\
\boldsymbol{f}^{\dot{l}}_{1,1}(r^*) & = & \dot{C}\sqrt{r^*} + \sqrt{2\dot{l}(\dot{l}+1)}r^*.
\end{array}
$$

Therefore, solutions of the radial part have the form

$$
\begin{aligned}
&\boldsymbol{f}^{l}_{1,1}(r)=\boldsymbol{f}^{l}_{1,-1}(r)=C\sqrt{r}+\sqrt{2l(l+1)}r,\\
&\boldsymbol{f}^{l}_{1,0}(r)=\sqrt{2l(l+1)}r,\\
&\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)=\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)=\dot{C}\sqrt{r^*}+\sqrt{2\dot{l}(\dot{l}+1)}r^*,\\
&\boldsymbol{f}^{\dot{l}}_{1,0}(r^*)=\sqrt{2\dot{l}(\dot{l}+1)}r^*.
\end{aligned}
$$

In such a way, solutions of the system (167) are defined by the following functions:

$$
\begin{aligned}
\phi_1(r,\varphi^c,\theta^c) &= \boldsymbol{f}^{l}_{1,1}(r)\mathfrak{M}^{l}_{1,n}(\varphi,\epsilon,\theta,\tau,0,0),\\
\phi_2(r,\varphi^c,\theta^c) &= \boldsymbol{f}^{l}_{1,0}(r)\mathfrak{M}^{l}_{0,n}(\varphi,\epsilon,\theta,\tau,0,0),\\
\phi_3(r,\varphi^c,\theta^c) &= \boldsymbol{f}^{l}_{1,-1}(r)\mathfrak{M}^{l}_{-1,n}(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\phi}_1(r^*,\dot{\varphi}^c,\dot{\theta}^c) &= \boldsymbol{f}^{\dot{l}}_{1,1}(r^*)\mathfrak{M}^{\dot{l}}_{1,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\phi}_2(r^*,\dot{\varphi}^c,\dot{\theta}^c) &= \boldsymbol{f}^{\dot{l}}_{1,0}(r^*)\mathfrak{M}^{\dot{l}}_{0,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),\\
\dot{\phi}_3(r^*,\dot{\varphi}^c,\dot{\theta}^c) &= \boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)\mathfrak{M}^{\dot{l}}_{-1,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),
\end{aligned}
$$

where

$$
\begin{aligned}
l=1,\ 2,\ 3,\ \ldots;\quad n=-l,\ -l+1,\ \ldots,\ l;\\
\dot{l}=1,\ 2,\ 3,\ \ldots;\quad \dot{n}=-\dot{l},\ -\dot{l}+1,\ \ldots,\ \dot{l},
\end{aligned}
$$

$$
\mathfrak{M}^{l}_{\pm1,n}(\varphi,\epsilon,\theta,\tau,0,0)=e^{\mp(\epsilon+i\varphi)}Z^{l}_{\pm1,n}(\theta,\tau),
$$

$$
Z^{l}_{\pm1,n}(\theta,\tau)=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l}i^{\pm1-k}\tan^{\pm1-k}\frac{\theta}{2}\tanh^{n-k}\frac{\tau}{2}\times
$$
$$
{}_2F_1\left(\begin{array}{c}\pm1-l+1,1-l-k\\ \pm1-k+1\end{array}\middle|\, i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{array}{c}n-l+1,1-l-k\\ n-k+1\end{array}\middle|\,\tanh^2\frac{\tau}{2}\right),
$$

$$
\mathfrak{M}^{l}_{0,n}(\varphi,\epsilon,\theta,\tau,0,0)=Z^{l}_{0,n}(\theta,\tau),
$$

$$
Z^{l}_{0,n}(\theta,\tau)=\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l}i^{-k}\tan^{-k}\frac{\theta}{2}\tanh^{n-k}\frac{\tau}{2}\times
$$
$$
{}_2F_1\left(\begin{array}{c}-l+1,1-l-k\\ -k+1\end{array}\middle|\, i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{array}{c}n-l+1,1-l-k\\ n-k+1\end{array}\middle|\,\tanh^2\frac{\tau}{2}\right),
$$

$$
\mathfrak{M}^{\dot{l}}_{\pm1,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0)=e^{\mp(\epsilon-i\varphi)}Z^{\dot{l}}_{\pm1,\dot{n}}(\theta,\tau),
$$

$$
Z^{\dot{l}}_{\pm1,\dot{n}}(\theta,\tau)=\cos^{2\dot{l}}\frac{\theta}{2}\cosh^{2\dot{l}}\frac{\tau}{2}\sum_{\dot{k}=-\dot{l}}^{\dot{l}}i^{\pm1-\dot{k}}\tan^{\pm1-\dot{k}}\frac{\theta}{2}\tanh^{\dot{n}-\dot{k}}\frac{\tau}{2}\times
$$
$$
{}_2F_1\left(\begin{array}{c}\pm1-\dot{l}+1,1-\dot{l}-\dot{k}\\ \pm1-\dot{k}+1\end{array}\middle|\, i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{array}{c}\dot{n}-\dot{l}+1,1-\dot{l}-\dot{k}\\ \dot{n}-\dot{k}+1\end{array}\middle|\,\tanh^2\frac{\tau}{2}\right),
$$

$$\mathfrak{M}^{\dot{l}}_{0,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0)=Z^{\dot{l}}_{0,\dot{n}}(\theta,\tau),$$

$$Z^{\dot{l}}_{0,\dot{n}}(\theta,\tau)=\cos^{2\dot{l}}\frac{\theta}{2}\cosh^{2\dot{l}}\frac{\tau}{2}\sum^{\dot{l}}_{\dot{k}=-\dot{l}}i^{-\dot{k}}\tan^{-\dot{k}}\frac{\theta}{2}\tanh^{\dot{n}-\dot{k}}\frac{\tau}{2}\times$$
$$ {}_2F_1\left(\begin{array}{c}-\dot{l}+1,1-\dot{l}-\dot{k}\\ -\dot{k}+1\end{array}\middle| i^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{array}{c}\dot{n}-\dot{l}+1,1-\dot{l}-\dot{k}\\ \dot{n}-\dot{k}+1\end{array}\middle|\tanh^2\frac{\tau}{2}\right).$$

Therefore, in accordance with the factorization (91), an explicit form of the particular solutions of the system (154)–(157) are given by expressions

$$\phi^l_{1,n}(\boldsymbol{\alpha})=\phi_+(\mathbf{k};\mathbf{x},t)\phi^l_{1,n}(\mathfrak{g})=$$
$$\left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\begin{pmatrix}\varepsilon_+(\mathbf{k})\\ \varepsilon_+(\mathbf{k})\end{pmatrix}\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]\boldsymbol{f}^l_{1,1}(r)\mathfrak{M}^l_{1,n}(\varphi,\epsilon,\theta,\tau,0,0),$$

$$\phi^l_{0,n}(\boldsymbol{\alpha})=\phi_0(\mathbf{k};\mathbf{x})\phi^l_{0,n}(\mathfrak{g})=\left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\begin{pmatrix}\varepsilon_0(\mathbf{k})\\ \varepsilon_0(\mathbf{k})\end{pmatrix}\exp[i\mathbf{k}\cdot\mathbf{x}]\boldsymbol{f}^l_{1,0}(r)\mathfrak{M}^l_{0,n}(0,0,\theta,\tau,0,0),$$

$$\phi^l_{-1,n}(\boldsymbol{\alpha})=\phi_-(\mathbf{k};\mathbf{x},t)\phi^l_{-1,n}(\mathfrak{g})=$$
$$\left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\begin{pmatrix}\varepsilon_-(\mathbf{k})\\ \varepsilon_-(\mathbf{k})\end{pmatrix}\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]\boldsymbol{f}^l_{1,-1}(r)\mathfrak{M}^l_{-1,n}(\varphi,\epsilon,\theta,\tau,0,0),$$

$$\dot{\phi}^{\dot{l}}_{1,\dot{n}}(\boldsymbol{\alpha})=\phi^*_+(\mathbf{k};\mathbf{x},t)\dot{\phi}^{\dot{l}}_{1,\dot{n}}(\mathfrak{g})=$$
$$\left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\begin{pmatrix}\varepsilon^*_+(\mathbf{k})\\ \varepsilon^*_+(\mathbf{k})\end{pmatrix}\exp[-i(\mathbf{k}\cdot\mathbf{x}-\omega t)]\boldsymbol{f}^{\dot{l}}_{1,1}(r^*)\mathfrak{M}^{\dot{l}}_{1,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),$$

$$\dot{\phi}^{\dot{l}}_{0,\dot{n}}(\boldsymbol{\alpha})=\phi^*_0(\mathbf{k};\mathbf{x})\dot{\phi}^{\dot{l}}_{0,\dot{n}}(\mathfrak{g})=\left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\begin{pmatrix}\varepsilon^*_0(\mathbf{k})\\ \varepsilon^*_0(\mathbf{k})\end{pmatrix}\exp[-i\mathbf{k}\cdot\mathbf{x}]\boldsymbol{f}^{\dot{l}}_{1,0}(r^*)\mathfrak{M}^{\dot{l}}_{0,\dot{n}}(0,0,\theta,\tau,0,0),$$

$$\dot{\phi}^{\dot{l}}_{-1,\dot{n}}(\boldsymbol{\alpha})=\phi^*_-(\mathbf{k};\mathbf{x},t)\dot{\phi}^{\dot{l}}_{-1,\dot{n}}(\mathfrak{g})=$$
$$\left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\begin{pmatrix}\varepsilon^*_-(\mathbf{k})\\ \varepsilon^*_-(\mathbf{k})\end{pmatrix}\exp[-i(\mathbf{k}\cdot\mathbf{x}-\omega t)]\boldsymbol{f}^{\dot{l}}_{1,-1}(r^*)\mathfrak{M}^{\dot{l}}_{-1,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0),\quad(170)$$

Before we proceed to define a general solution of the boundary value problem, let us note the following circumstance. The set (170) consists of the transverse solutions $\phi^l_{\pm1,n}(\boldsymbol{\alpha})$ (negative helicity), $\dot{\phi}^{\dot{l}}_{\pm1,\dot{n}}(\boldsymbol{\alpha})$ (positive helicity) and the zero-eigenvalue (longitudinal) solutions $\phi^l_{0,n}(\boldsymbol{\alpha})$ and $\dot{\phi}^{\dot{l}}_{0,\dot{n}}(\boldsymbol{\alpha})$. The longitudinal solutions $\phi^l_{0,n}(\boldsymbol{\alpha})$ and $\dot{\phi}^{\dot{l}}_{0,\dot{n}}(\boldsymbol{\alpha})$ do not contribute to a real photon due to their transversality conditions (162) and (164). Thus, any real photon should be described by only $\phi^l_{\pm1,n}(\boldsymbol{\alpha})$ and $\dot{\phi}^{\dot{l}}_{\pm1,\dot{n}}(\boldsymbol{\alpha})$:

$$\phi^l_{\pm1,n}(\boldsymbol{\alpha})=\phi_\pm(\mathbf{k};\mathbf{x},t)\phi^l_{\pm1,n}(\mathfrak{g})=$$
$$\left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\begin{pmatrix}\varepsilon_\pm(\mathbf{k})\\ \varepsilon_\pm(\mathbf{k})\end{pmatrix}\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]\boldsymbol{f}^l_{1,\pm1}(r)\mathfrak{M}^l_{\pm1,n}(\varphi,\epsilon,\theta,\tau,0,0),$$

$$\dot{\phi}^{\dot{l}}_{\pm 1,\dot{n}}(\boldsymbol{\alpha}) = \phi^*_{\pm}(\mathbf{k};\mathbf{x},t)\dot{\phi}^{\dot{l}}_{\pm 1,\dot{n}}(\mathfrak{g}) = \left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\begin{pmatrix}\varepsilon^*_{\pm}(\mathbf{k})\\ \varepsilon^*_{\pm}(\mathbf{k})\end{pmatrix}\exp[-i(\mathbf{k}\cdot\mathbf{x}-\omega t)]\boldsymbol{f}^{\dot{l}}_{1,\pm 1}(r^*)\mathfrak{M}^{\dot{l}}_{\pm 1,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0).$$

Taking into account only these physically meaningful solutions, we can define a general solution of the boundary value problem by means of the following expansions

$$\boldsymbol{\phi}_{\pm 1}(\boldsymbol{\alpha}) = \left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\sum_k\begin{pmatrix}\varepsilon_{\pm}(\mathbf{k})\\ \varepsilon_{\pm}(\mathbf{k})\end{pmatrix}e^{ikx}\sum_{l=1}^{\infty}\boldsymbol{f}^l_{1,\pm 1}(r)\sum_{n=-l}^{l}\alpha^{\pm 1}_{l,n}\mathfrak{M}^l_{\pm 1,n}(\varphi,\epsilon,\theta,\tau,0,0), \quad (171)$$

$$\dot{\boldsymbol{\phi}}_{\pm 1}(\boldsymbol{\alpha}) = \left\{2(2\pi)^3\right\}^{-\frac{1}{2}}\sum_k\begin{pmatrix}\varepsilon^*_{\pm}(\mathbf{k})\\ \varepsilon^*_{\pm}(\mathbf{k})\end{pmatrix}e^{-ikx}\sum_{\dot{l}=1}^{\infty}\boldsymbol{f}^{\dot{l}}_{1,\pm 1}(r^*)\sum_{\dot{n}=-\dot{l}}^{\dot{l}}\alpha^{\pm 1}_{\dot{l},\dot{n}}\mathfrak{M}^{\dot{l}}_{\pm 1,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0) \quad (172)$$

where

$$\alpha^{\pm 1}_{l,n} = \frac{(-1)^n(2l+1)(2\dot{l}+1)}{32\pi^4\boldsymbol{f}^l_{1,\pm 1}(a)}\int\limits_{\mathbb{S}^2}\int\limits_{T_4}F_{\pm 1}(\boldsymbol{\alpha})e^{-ikx}\mathfrak{M}^l_{\pm 1,n}(\varphi,\epsilon,\theta,\tau,0,0)d^4xd^4\mathfrak{g},$$

$$\alpha^{\pm 1}_{\dot{l},\dot{n}} = \frac{(-1)^{\dot{n}}(2l+1)(2\dot{l}+1)}{32\pi^4\boldsymbol{f}^{\dot{l}}_{1,\pm 1}(a^*)}\int\limits_{\mathbb{S}^2}\int\limits_{T_4}\dot{F}_{\pm 1}(\boldsymbol{\alpha})e^{ikx}\mathfrak{M}^{\dot{l}}_{\pm 1,\dot{n}}(\varphi,\epsilon,\theta,\tau,0,0)d^4xd^4\mathfrak{g}.$$

9.1 Quantization

In case of the photon field we define the field operator by a following superposition

$$\boldsymbol{\phi}(\boldsymbol{\alpha}) = \sum_{s=1}^{2}\mathsf{c}_s\phi_s(\boldsymbol{\alpha}) + \sum_{s=1}^{2}\mathsf{c}^+_s\dot{\phi}_s(\boldsymbol{\alpha}), \quad (173)$$

where c^+_s and c_s are creation (emission) and annihilation (absorption) operators of the photon in a state s, $\phi_s(\boldsymbol{\alpha})$ and $\dot{\phi}_s(\boldsymbol{\alpha})$ are Fourier series (171) (negative helicity) and (172) (positive helicity), here we designate $\phi_1(\boldsymbol{\alpha}) = \boldsymbol{\phi}_1(\boldsymbol{\alpha})$, $\phi_2(\boldsymbol{\alpha}) = \boldsymbol{\phi}_{-1}(\boldsymbol{\alpha})$, $\dot{\phi}_1(\boldsymbol{\alpha}) = \dot{\boldsymbol{\phi}}_1(\boldsymbol{\alpha})$, $\dot{\phi}_2(\boldsymbol{\alpha}) = \dot{\boldsymbol{\phi}}_{-1}(\boldsymbol{\alpha})$. The operators c^+_s and c_s satisfy the following commutation relations:

$$\left[\mathsf{c}_s, \mathsf{c}^+_{s'}\right]_- = \delta_{ss'},$$
$$\left[\mathsf{c}_s, \mathsf{c}_{s'}\right]_- = \left[\mathsf{c}^+_s, \mathsf{c}^+_{s'}\right]_- = 0, \quad s, s' = 1, 2.$$

Taking into account the latter relations, let us calculate a commutator of the photon field:

$$\left[\boldsymbol{\phi}_\alpha(\boldsymbol{\alpha}), \boldsymbol{\phi}_\beta(\boldsymbol{\alpha}')\right]_- = D_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}'), \quad (174)$$

where

$$D_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}') = D^+_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}') + D^-_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}'),$$
$$D^+_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}') = \sum_{s=1}^{2}\phi_{s\alpha}(\boldsymbol{\alpha})\dot{\phi}_{s\beta}(\boldsymbol{\alpha}'),$$
$$D^-_{\alpha\beta}(\boldsymbol{\alpha},\boldsymbol{\alpha}') = \sum_{s=1}^{2}\dot{\phi}_{s\alpha}(\boldsymbol{\alpha})\phi_{s\beta}(\boldsymbol{\alpha}').$$

Or, using the formulae (171) and (172), we find an explicit form of the functions $D^{+}_{\alpha\beta}(\boldsymbol{\alpha}-\boldsymbol{\alpha}')$ and $D^{-}_{\alpha\beta}(\boldsymbol{\alpha}-\boldsymbol{\alpha}')$:

$$D^{+}_{\alpha\beta}(\boldsymbol{\alpha}-\boldsymbol{\alpha}') = \left\{2(2\pi)^3\right\}^{-1}\sum_{s=1}^{2}\sum_{k}\boldsymbol{\varepsilon}_{\alpha s}(\mathbf{k})\boldsymbol{\varepsilon}^{*}_{\beta s}(\mathbf{k})e^{ik(x-x')}\times$$
$$e^{-i(-1)^{s-1}(\varphi^{c}-\varphi^{c\prime})}\sum_{l=1}^{\infty}\boldsymbol{f}^{l}_{1,(-1)^{s-1}}(r)\boldsymbol{f}^{l}_{1,(-1)^{s-1}}(r^{*})Z^{l}_{(-1)^{s-1},(-1)^{s-1}}(\cos\theta^{c\prime\prime}),$$

$$D^{-}_{\alpha\beta}(\boldsymbol{\alpha}-\boldsymbol{\alpha}') = \left\{2(2\pi)^3\right\}^{-1}\sum_{s=1}^{2}\sum_{k}\boldsymbol{\varepsilon}^{*}_{\alpha s}(\mathbf{k})\boldsymbol{\varepsilon}_{\beta s}(\mathbf{k})e^{-ik(x-x')}\times$$
$$e^{i(-1)^{s-1}(\varphi^{c}-\varphi^{c\prime})}\sum_{l=1}^{\infty}\boldsymbol{f}^{l}_{1,(-1)^{s-1}}(r^{*})\boldsymbol{f}^{l}_{1,(-1)^{s-1}}(r)Z^{l}_{(-1)^{s-1},(-1)^{s-1}}(\cos\theta^{c\prime\prime}).$$

Let us define now normal and time ordered products of the field operators (173). Supposing

$$\boldsymbol{\phi}(\boldsymbol{\alpha}) = \phi^{(+)}(\boldsymbol{\alpha}) + \phi^{(-)}(\boldsymbol{\alpha}),$$

where $\phi^{(+)}(\boldsymbol{\alpha})$ and $\phi^{(-)}(\boldsymbol{\alpha})$ are annihilation and creation operators of the photons, we see that

$$\begin{aligned}
N(\phi^{(+)}(\boldsymbol{\alpha})\phi^{(-)}(\boldsymbol{\alpha}')) &= \phi^{(-)}(\boldsymbol{\alpha}')\phi^{(+)}(\boldsymbol{\alpha}),\\
N(\phi^{(-)}(\boldsymbol{\alpha})\phi^{(+)}(\boldsymbol{\alpha}')) &= \phi^{(-)}(\boldsymbol{\alpha})\phi^{(+)}(\boldsymbol{\alpha}'),\\
N(\phi^{(-)}(\boldsymbol{\alpha})\phi^{(-)}(\boldsymbol{\alpha}')) &= \phi^{(-)}(\boldsymbol{\alpha})\phi^{(-)}(\boldsymbol{\alpha}'),\\
N(\phi^{(+)}(\boldsymbol{\alpha})\phi^{(+)}(\boldsymbol{\alpha}')) &= \phi^{(+)}(\boldsymbol{\alpha})\phi^{(+)}(\boldsymbol{\alpha}').
\end{aligned}$$

Taking into account the latter N-products, we have

$$\begin{aligned}
\boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}(\boldsymbol{\alpha}') &= (\phi^{(+)}(\boldsymbol{\alpha}) + \phi^{(-)}(\boldsymbol{\alpha}))(\phi^{(+)}(\boldsymbol{\alpha}') + \phi^{(-)}(\boldsymbol{\alpha}')) =\\
&= \phi^{(+)}(\boldsymbol{\alpha})\phi^{(+)}(\boldsymbol{\alpha}') + \phi^{(-)}(\boldsymbol{\alpha})\phi^{(+)}(\boldsymbol{\alpha}') + \phi^{(+)}(\boldsymbol{\alpha})\phi^{(-)}(\boldsymbol{\alpha}') +\\
&\quad + \phi^{(-)}(\boldsymbol{\alpha})\phi^{(-)}(\boldsymbol{\alpha}').
\end{aligned}$$

In accordance with (174) $\phi^{(+)}(\boldsymbol{\alpha})\phi^{(-)}(\boldsymbol{\alpha}') = \phi^{(-)}(\boldsymbol{\alpha}')\phi^{(+)}(\boldsymbol{\alpha}) + D^{-}(\boldsymbol{\alpha}-\boldsymbol{\alpha}')$, therefore,

$$\begin{aligned}
\boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}(\boldsymbol{\alpha}') &= N(\boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}(\boldsymbol{\alpha}')) + D^{-}(\boldsymbol{\alpha}-\boldsymbol{\alpha}')\\
&= N(\boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}(\boldsymbol{\alpha}')) + \underline{\boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}}(\boldsymbol{\alpha}'),
\end{aligned}$$

where

$$\underline{\boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}}(\boldsymbol{\alpha}') = \langle 0 \mid \boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}(\boldsymbol{\alpha}') \mid 0\rangle = D^{-}(\boldsymbol{\alpha}-\boldsymbol{\alpha}')$$

is an operator coupling of the photon fields.

Further, the time ordered coupling of the field operators (173) is defined as

$$\begin{aligned}
\overline{\boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}}(\boldsymbol{\alpha}') &= \langle 0 \mid T(\boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}(\boldsymbol{\alpha}')) \mid 0\rangle =\\
&= \begin{cases} \langle 0 \mid \boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}(\boldsymbol{\alpha}') \mid 0\rangle = \underline{\boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}}(\boldsymbol{\alpha}') = D^{-}(\boldsymbol{\alpha}-\boldsymbol{\alpha}'), & t > t';\\ \langle 0 \mid \boldsymbol{\phi}(\boldsymbol{\alpha}')\boldsymbol{\phi}(\boldsymbol{\alpha}) \mid 0\rangle = \underline{\boldsymbol{\phi}(\boldsymbol{\alpha}')\boldsymbol{\phi}}(\boldsymbol{\alpha}) = D^{+}(\boldsymbol{\alpha}-\boldsymbol{\alpha}'), & t' > t. \end{cases}
\end{aligned}$$

10 Interacting fields

Up to now we analyze free Dirac and Maxwell fields. Let us consider an interaction between these fields. As usual, interactions between the fields are described by an interaction Lagrangian $\mathcal{L}_I$. In our case we take the following Lagrangian

$$\mathcal{L}_I(\boldsymbol{\alpha}) = \mu(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\sigma^D_{(\mu\nu)^k}\boldsymbol{\psi}(\boldsymbol{\alpha}))(\Xi^M_{(\rho)k}\boldsymbol{\phi}(\boldsymbol{\alpha})), \tag{175}$$

where $\sigma^D_{\mu\nu} = \frac{1}{2}(\Xi^D_\mu\Xi^D_\nu - \Xi^D_\nu\Xi^D_\mu)$ and $\Xi^D = (\Gamma^D_0, \Gamma^D_1, \Gamma^D_2, \Gamma^D_3, \Upsilon^D_1, \Upsilon^D_2, \Upsilon^D_3, \Upsilon^D_4, \Upsilon^D_5, \Upsilon^D_6)$, $\Xi^M = (\Gamma^M_0, \Gamma^M_1, \Gamma^M_2, \Gamma^M_3, \Upsilon^M_1, \Upsilon^M_2, \Upsilon^M_3, \Upsilon^M_4, \Upsilon^M_5, \Upsilon^M_6)$, here Γ^D and Υ^D are the matrices (123) and (124)–(125), and Γ^M and Υ^M are the matrices (148) and (149)–(150).

The full Lagrangian of interacting Dirac and Maxwell fields equals to a sum of the free field Lagrangians and the interaction Lagrangian:

$$\mathcal{L}(\boldsymbol{\alpha}) = \mathcal{L}_D(\boldsymbol{\alpha}) + \mathcal{L}_M(\boldsymbol{\alpha}) + \mathcal{L}_I(\boldsymbol{\alpha}),$$

where $\mathcal{L}_D(\boldsymbol{\alpha})$ and $\mathcal{L}_M(\boldsymbol{\alpha})$ are of the type (122) and (147), respectively. Or,

$$\begin{aligned}\mathcal{L}(\boldsymbol{\alpha}) = -\frac{1}{2}\left(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\Xi^D_\mu\frac{\partial\boldsymbol{\psi}(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}_\mu} - \frac{\partial\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}_\mu}\Xi^D_\mu\boldsymbol{\psi}(\boldsymbol{\alpha})\right) - \\ -\frac{1}{2}\left(\overline{\boldsymbol{\phi}}(\boldsymbol{\alpha})\Xi^M_\mu\frac{\partial\boldsymbol{\phi}(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}_\mu} - \frac{\partial\overline{\boldsymbol{\phi}}(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}_\mu}\Xi^M_\mu\boldsymbol{\phi}(\boldsymbol{\alpha})\right) - \\ -\kappa\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\boldsymbol{\psi}(\boldsymbol{\alpha}) + \mu(\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})\sigma^D_{(\mu\nu)^k}\boldsymbol{\psi}(\boldsymbol{\alpha}))(\Xi^M_{(\rho)k}\boldsymbol{\phi}(\boldsymbol{\alpha})).\end{aligned}$$

Since the Lagrangian (175) does not contain derivatives on the field functions, then for a Hamiltonian density we have $\mathcal{H}_I(\boldsymbol{\alpha}) = -\mathcal{L}_I(\boldsymbol{\alpha})$.

As is known, in the standard quantum field theory the S-matrix is expressed via the Dyson formula [95]

$$S = T\left[\exp\left(-\frac{i}{\hbar c}\int\limits_{-\infty}^{+\infty}\mathcal{H}_I(x)d^4x\right)\right], \tag{176}$$

where T is the time ordering operator. The Hamiltonian density $\mathcal{H}_I(x)$ has in general been assumed to have the form of an invariant local products of fields.

In our case, the electron-positron and photon fields are defined on the space $\mathcal{M}_8 = \mathbb{R}^{1,3}\times\mathbb{S}^2$ which larger then the Minkowski space $\mathbb{R}^{1,3}$. With a view to define a formula similar to the equation (176) it is necessarily to replace d^4x by the following invariant measure on $\mathcal{M}_8$:

$$d^8\mu = d^4x d^4\mathfrak{g},$$

where

$$d^4\mathfrak{g} = \sin\theta^c\sin\dot{\theta}^c d\theta d\tau d\varphi d\epsilon.$$

Therefore, an analog of the Dyson formula (176) on the manifold $\mathcal{M}_8$ can be written as follows

$$S = T\left[\exp\left(-\frac{i}{\hbar c}\int\limits_{T_4}\int\limits_{\mathbb{S}^2}\mathcal{H}_I(\boldsymbol{\alpha})d^4x d^4\mathfrak{g}\right)\right].$$

Concrete calculations of the scattering amplitudes come beyond the framework of the present paper and will be considered in a future work.

11 Conclusion

In this paper we have presented a general scheme of construction of quantum electrodynamics on the Poincaré group $\mathcal{P}$ (or, equally, on the homogeneous spaces of $\mathcal{P}$). Except the maximal homogeneous space $\mathcal{M}_{10}$, we consider here only three spaces $\mathcal{M}_8$, $\mathcal{M}_7$ and $\mathcal{M}_6$ (it is easy to see that $\mathcal{M}_8$, $\mathcal{M}_7$ and $\mathcal{M}_6$ correspond to the Wigner's little groups $SL(2,\mathbb{C})$, $SU(1,1)$ and $SU(2)$, respectively). However, the similar constructions are possible for all other homogeneous spaces of $\mathcal{P}$ contained in the Finkelstein-Bacry-Kihlberg list [32, 5] and endowed with a measure.

It should be noted that discrete symmetries remain outside of our consideration. It is well-known that discrete transformations are of fundamental importance for constructing relativistic wave equations and for their analysis. An inclusion of discrete symmetries into the framework of quantum field theory on the Poincaré group can be obtained via an automorphism representation. It is known that Gel'fand, Minlos and Shapiro [36] proposed to consider the discrete transformations as outer involutory automorphisms of the Lorentz group (there are also other realizations of the discrete symmetries via the outer automorphisms, see [74, 62, 103]). At present, the Gel'fand-Minlos-Shapiro ideas have been found further development in the works of Buchbinder, Gitman and Shelepin [16, 42], where the discrete symmetries are represented by both outer and inner automorphisms of the Poincaré group. It is pointed out by Shirokov [101, 102] that an universal covering of the inhomogeneous Lorentz group has eight inequivalent realizations. Later on, in the eighties this idea was applied to a general orthogonal group $O(p,q)$ by Dąbrowski [21]. It is well-known that universal coverings of the groups $O(p,q)$ are completely formulated within spinor groups [68]. In turn, the spinor group is an intrinsic notion of the Clifford algebra [17, 68]. By this reason there exists a complete and consistent description of the discrete transformations in terms of the Clifford algebra theory. Such a description has been given in the works [112, 113, 116], where the discrete symmetries are represented by fundamental automorphisms of the Clifford algebras. The fundamental automorphisms are compared to elements of the finite group formed by the discrete transformations. In like manner, charge conjugation is naturally included into a general scheme by means of a complex conjugation pseudoautomorphism [88, 89, 116]. It allows us to define 64 universal coverings of $O(p,q)$ (CPT-structures) which include as a particular case the eight PT-structures of Shirokov and Dąbrowski [116]. Moreover, it allows us to incorporate this algebraic description with the ideas presented in [16] and then to apply it for analysis of relativistic wave equations on the homogeneous spaces of the Poincaré group.

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